Solutions for PSet 6

1. (8.22:14)

(a) \( f(x, y) = f_1(x, y)i + f_2(x, y)j \), where \( f_1(x, y) = e^{x+2y} \), and \( f_2(x, y) = \sin(y + 2x) \). Computing all the partial derivatives

\[
\begin{align*}
\frac{\partial f_1}{\partial x} &= e^{x+2y} \\
\frac{\partial f_1}{\partial y} &= 2e^{x+2y} \\
\frac{\partial f_2}{\partial x} &= 2\cos(y + 2x) \\
\frac{\partial f_2}{\partial y} &= \cos(y + 2x)
\end{align*}
\]

So the matrix for the total derivative is:

\[
Df(x, y) = \begin{pmatrix}
e^{x+2y} & 2e^{x+2y} \\
2\cos(y + 2x) & \cos(y + 2x)
\end{pmatrix}
\]

Similarly for \( g(u, v, w) = g_1(u, v, w)i + g_2(u, v, w)j \), where \( g_1(u, v, w) = u + 2v^2 + 3w^3 \) and \( g_2(u, v, w) = 2v - u^2 \) we have:

\[
\begin{align*}
\frac{\partial g_1}{\partial u} &= 1 \\
\frac{\partial g_1}{\partial v} &= 4v \\
\frac{\partial g_1}{\partial w} &= 9w^2 \\
\frac{\partial g_2}{\partial u} &= -2u \\
\frac{\partial g_2}{\partial v} &= 2 \\
\frac{\partial g_2}{\partial w} &= 0
\end{align*}
\]

And the total derivative is:

\[
Dg(u, v, w) = \begin{pmatrix}
1 & 4v & 9w^2 \\
-2u & 2 & 0
\end{pmatrix}
\]

(b) The composition

\[
h(u, v, w) = f(g(u, v, w)) = \exp(u+2v^2+3w^3+4v-2u^2)i + \sin(2v-u^2+2u+4v^2+6w^3)j
\]

(c) The total derivative at a point \((u, v, w)\) can be computed using the chain rule:

\[
Dh(u, v, w) = Df(g(u, v, w))Dg(u, v, w)
\]

\[
= \begin{pmatrix}
e^{g_1+2g_2} & 2e^{g_1+2g_2} \\
2\cos(g_2 + 2g_1) & \cos(g_2 + 2g_1)
\end{pmatrix}
\begin{pmatrix}
1 & 4v & 9w^2 \\
-2u & 2 & 0
\end{pmatrix}
\]

Now we evaluate at \((u, v, w) = (1, -1, 1)\) and thus \(g_1 = 6, g_2 = -3\). As a result \(g_1 + 2g_2 = 0\) and \(g_2 + 2g_1 = 9\) and

\[
Dh(1, -1, 1) = \begin{pmatrix}
e^0 & 2e^0 \\
2\cos 9 & \cos 9
\end{pmatrix}
\begin{pmatrix}
1 & -4 & 9 \\
-2 & 2 & 0
\end{pmatrix} =
\begin{pmatrix}
-3 & 0 & 9 \\
0 & -6\cos 9 & 18\cos 9
\end{pmatrix}
\]
2. (8.24:12)

(a) We can compute $\nabla (\frac{1}{r})$ using the chain rule for the functions $r : \mathbb{R}^3 \to \mathbb{R}$ defined by $r(r) = \sqrt{r \cdot r}$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = \frac{1}{t}$. With these functions $\frac{1}{r} = g \circ r$ thus

$$A \nabla \left( \frac{1}{r} \right) = A \cdot g'(r) \nabla \left( \sqrt{r \cdot r} \right) = A \cdot -\frac{1}{(r^2)^{\frac{3}{2}}} \cdot \frac{2r}{2 \sqrt{r \cdot r}} = -\frac{A}{r^3} \cdot r$$

(b) To evaluate the left hand side in question, we need to first evaluate $\nabla \left( A \nabla \left( \frac{1}{r} \right) \right)$. Using part (a), this is equivalent to

$$\nabla \left( -\frac{A \cdot r}{r^3} \right) = \nabla \left( \frac{f(r)}{h(r)} \right)$$

where $f(r) = -A \cdot r$ and $h(r) = r^3$. Both are real-valued functions, thus we can apply the rule for their fractions:

$$\nabla \left( -\frac{A \cdot r}{r^3} \right) = \nabla \left( \frac{f(r)}{h(r)} \right) = \frac{\nabla(f(r))h(r) - f(r) \nabla(h(r))}{(h(r))^2}$$

But

$$\nabla h(r) = \nabla r^3 = \nabla (r \cdot r)^{\frac{3}{2}} = \frac{3}{2} (r \cdot r)^{\frac{3}{2}} 2r = 3rr$$

Therefore

$$\nabla \left( -\frac{A \cdot r}{r^3} \right) = \frac{Ar^3 - A \cdot r3rr}{r^6}$$

Now

$$B \cdot \nabla \left( -\frac{A \cdot r}{r^3} \right) = \frac{3B \cdot rA \cdot r}{r^5} - \frac{A \cdot B}{r^3}$$

3. To compute the gradient of multivariate function $f(x, y)$, compute the partial derivatives:

$$\frac{\partial f(x, y)}{\partial x} = \lim_{h \to 0} \frac{\int_0^{(x+h)y} g(u)du - \int_0^{xy} g(u)du}{h}$$

$$= \lim_{h \to 0} \frac{\int_{xy}^{(x+h)y} g(u)du}{h}$$
If we define a function \( m(h) = \int_{xy}^{xy+hy} g(u)du \) then the limit above is \( m'(h) \). Using the fundamental theorem of calculus, we determine \( m'(0) = yg(xy) \). The partial derivative with respect to \( y \) follows similarly. Thus the gradient of \( f(x, y) = \int_0^{xy} g(u)du \) is \( \nabla f(x, y) = \langle yg(xy), xg(xy) \rangle \).

A level set \((x, y)\) can be described as \( f^{-1}(c) \). If both \((x_0, y_0)\) and \((x, y)\) lie in the same level set, then:

\[
\int_0^{xy} g(u)du = \int_0^{x_0y_0} g(u)du = c
\]

Or,

\[
\int_{x_0y_0}^{xy} g(u)du = 0
\]

As \( g \) is a positive function, its integral can only be 0 if the integration interval is empty or:

\[
xy = x_0y_0 \quad \text{or} \quad \exists \text{ a value } b \text{ s.t. } xy = x_0y_0 = b
\]

As \( g \) is positive, the function \( m(t) = \int_0^t g(u)du \) is strictly increasing in \( t \).

Therefore, there exists a unique \( b \) such that \( \int_0^b g(u)du = c \neq 0 \). In other words, the level set is parametrized by \( y = h(x) = \frac{b}{x} \) where \( b \) is unique.

A level set \((x, y)\) can be parametrized as \( r(x) = f^{-1}(c) = (x, \frac{b}{x}) \). This level set has slope \( r'(x) \) given by \( (1, -\frac{b}{x^2}) \). The gradient of \( f(x, y) \), \( \nabla f \) at any point \((x, y)\) is \( g(b)(\frac{b}{x}, x) \). The dot product \( \nabla f \cdot r'(x) \) is:

\[
g(b)(\frac{b}{x}, x) \cdot (1, -\frac{b}{x^2}) = g(b)\frac{b}{x} - g(b)\frac{bx}{x^2} = 0
\]

Hence, \( \nabla f \) is orthogonal to the level set at each point on the curve.

4.

\[
f(x, y) = \begin{cases} 
x^2 - \frac{y^2}{x+y}xy & \text{if } (x, y) \neq (0,0), \\
0 & \text{if } (x, y) = (0,0). 
\end{cases}
\]
The partial derivatives:

\[
\frac{\partial f}{\partial x}(0, y) = \lim_{h \to 0} \frac{f(h, y) - f(0, y)}{h} = \lim_{h \to 0} \frac{h^2 - y^2}{h^2 + y^2} = -y
\]

\[
\frac{\partial f}{\partial y}(x, 0) = \lim_{h \to 0} \frac{f(x, h) - f(x, 0)}{h} = \lim_{h \to 0} \frac{x^2 - h^2}{x^2 + h^2} = x
\]

Using the above derivation, the second partial derivatives can be evaluated at point \((0,0)\):

\[
\frac{\partial^2 f}{\partial x \partial y}(0, 0) = \frac{\partial}{\partial x} \left. \frac{\partial f}{\partial y}(x, 0) \right|_{(0,0)} = \left. \frac{\partial}{\partial x} (x) \right|_{(0,0)} = 1
\]

and

\[
\frac{\partial^2 f}{\partial y \partial x}(0, 0) = \frac{\partial}{\partial y} \left. \frac{\partial f}{\partial x}(0, y) \right|_{(0,0)} = \left. \frac{\partial}{\partial y} (-y) \right|_{(0,0)} = -1
\]

Thus

\[
\frac{\partial^2 f}{\partial x \partial y}(0, 0) \neq \frac{\partial^2 f}{\partial y \partial x}(0, 0)
\]

This means that in general,

\[
\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}
\]

5. We can write \(F(t) = f(r(t))\) where \(r(t) = (3t^2, 2t, 1, 3 - t^3)\).

Then \(F(t) = f \circ r\), thus

\[
F'(t) = \nabla f(3t^2, 2t, 1, 3 - t^3) \cdot r'(t) = \nabla f(3t^2, 2t, 1, 3 - t^3) \cdot (6t, 2, -3t^2)
\]

At \(t = 1\) this evaluates to:

\[
F'(1) = \nabla f(3, 3, 2) \cdot (6, 2, -3)
\]

The gradient of \(f : \mathbb{R}^3 \to \mathbb{R}\) is \(\nabla f(x, y, z) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})\). Thus

\[
F'(1) = 6 \frac{\partial f}{\partial x}(3, 3, 2) + 2 \frac{\partial f}{\partial y}(3, 3, 2) - 3 \frac{\partial f}{\partial z}(3, 3, 2)
\]
Let $Hess_f$ denote the second derivative matrix of $f$:

$$Hess_f(x, y, z) = \begin{pmatrix}
\frac{\partial^2 f}{\partial x^2}(x, y, z) & \frac{\partial^2 f}{\partial x \partial y}(x, y, z) & \frac{\partial^2 f}{\partial x \partial z}(x, y, z) \\
\frac{\partial^2 f}{\partial y \partial x}(x, y, z) & \frac{\partial^2 f}{\partial y^2}(x, y, z) & \frac{\partial^2 f}{\partial y \partial z}(x, y, z) \\
\frac{\partial^2 f}{\partial z \partial x}(x, y, z) & \frac{\partial^2 f}{\partial z \partial y}(x, y, z) & \frac{\partial^2 f}{\partial z^2}(x, y, z)
\end{pmatrix}$$

Then

$$F''(t) = r'(t)Hess_f(3t^2, 2t + 1, 3 - t^3) + \nabla f(3t^2, 2t + 1, 3 - t^3) \cdot r''(t)$$

$$= (6t, 2, -3t^2)f''(3t^2, 2t + 1, 3 - t^3)(6t, 2, -3t^2)^T + \nabla f(3t^2, 2t + 1, 3 - t^3) \cdot (6, 0, -6t)$$

$$= 36t^2\frac{\partial^2 f}{\partial x^2} + 12t \frac{\partial^2 f}{\partial x \partial y} - 18t^3 \frac{\partial^2 f}{\partial x \partial z} + 12t \frac{\partial^2 f}{\partial y \partial x} + 4 \frac{\partial^2 f}{\partial y^2} - 6t^2 \frac{\partial^2 f}{\partial y \partial z} - 18t^3 \frac{\partial^2 f}{\partial z \partial x}$$

$$- 6t^2 \frac{\partial^2 f}{\partial z \partial y} + 9t^4 \frac{\partial^2 f}{\partial z^2} + 6 \frac{\partial f}{\partial x} - 6t \frac{\partial f}{\partial z}$$

where all partial derivatives are taken at $(3t^2, 2t + 1, 3 - t^3)$.

Substituting $t = 1$ we get

$$F''(t) = 36 \frac{\partial^2 f}{\partial x^2} + 12 \frac{\partial^2 f}{\partial x \partial y} - 18 \frac{\partial^2 f}{\partial x \partial z} + 12 \frac{\partial^2 f}{\partial y \partial x} + 4 \frac{\partial^2 f}{\partial y^2} - 6 \frac{\partial^2 f}{\partial y \partial z} - 18 \frac{\partial^2 f}{\partial z \partial x}$$

$$- 6 \frac{\partial^2 f}{\partial z \partial y} + 9 \frac{\partial^2 f}{\partial z^2} + 6 \frac{\partial f}{\partial x} - 6 \frac{\partial f}{\partial z}$$

where the partial derivatives are taken at $(3, 3, 2)$.

6. (a) $h(x) = g(f(x))$. Thus $Dh((0, 0)) =$

$$Dg(f((0, 0)))Df(0, 0) = Dg(1, 2)Df(0, 0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 6 & 3 \\ 4 & 7 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 6 & 3 \\ 4 & 7 \end{pmatrix}$$

(b) Let $k = f^{-1}$, then $Dk(0, 0) =$

$$(Df(f^{-1}(0, 0)))^{-1} = (Df(1, 2))^{-1}$$

Thus,

$$Dk(0, 0) = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$$