18.075 Practice Test II for Exam 2

Justify your answers. Cross out what is not meant to be part of your solution.

Total number of points: 60.

I. Consider the integral

\[ I = \int_{-\infty}^{\infty} \frac{x \sin x}{(x^2 - \pi^2)(x^2 + 1)} \, dx \quad (1). \]

1. (2pts) By replacing \( x \) by the complex variable \( z \), locate and characterize all
   singularities of the integrand (viewed as a function of \( z \)).

2. (2pts) Locate and characterize all the singularities of \( \frac{z e^{iz}}{(x^2 - \pi^2)(x^2 + 1)} \), i.e., repeat
   part (a) after replacing \( \sin x = \text{Im}(e^{ix}) \) by \( e^{iz} \).

3. (6pts) Define the principal value and indented contours in order to evaluate
   the given integral \( I \) of (1) above. In particular, state which parts of the contours
   finally contribute zero and why.

4. (10pts) Evaluate the requisite integral \( I \) by use of the residue theorem.

II. (15pts) Consider the real integral

\[ I = \int_{0}^{\infty} \frac{x}{x^4 + 1} \, dx. \]

Evaluate \( I \) by use of the residue theorem. **Hint:** Integrate \( f(z) = \frac{z}{i(z^4 - 1)} \)
(\( z = x + iy \)) around the closed contour consisting of the portions of the \( x \) (real) and \( y \)
(imaginary) axes for which \( 0 \leq x \leq R \), and \( 0 \leq y \leq R \), and a quadrant of the circle
\(|z| = R \), and finally let \( R \to \infty \).

III. Find the region of convergence of the following series by using the ratio or
Cauchy (root) test, where \( x \) is real.

1. (4pts) \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \).

2. (4pts) \( \sum_{n=0}^{\infty} \frac{x^n}{n!} (x + 2)^n \).

IV. Locate and classify the singular points of the following differential equations.

1. (2pts) \( \frac{d^2 u}{dy^2} + x \frac{du}{dx} + x^2 y = 0. \)

2. (3pts) \( x^2 \frac{d^2 y}{dx^2} - (x^2 + 2) \frac{dy}{dx} - (x + 1)y = 0. \)

V. Consider the differential equation

\[ x^2 \frac{d^2 y}{dx^2} + (x^2 - x) \frac{dy}{dx} + y = 0. \]

1. (6pts) By substituting \( y = \sum_{n=0}^{\infty} A_n x^n \), express the left-hand side of the differential equation as a power series, each term involving the (common) factor \( x^n \).

2. (6pts) Determine the recurrence formula for the coefficients \( A_n \). (You are NOT asked to find the final solution \( y(x) \).) How many independent solutions does this method give?