1) (30 pts.)  

(a) Solve this cyclic convolution equation for the vector \( d \). (I would transform convolution to multiplication.) Notice that \( c = (5, 0, 0, 0) - (1, 1, 1, 1) \). The equation is like deconvolution.

\[
    c \otimes d = (4, -1, -1, -1) \otimes (d_0, d_1, d_2, d_3) = (1, 0, 0, 0).
\]

(b) Why is there no solution \( d \) if I change \( c \) to \( C = (3, -1, -1, -1) \)? Try it. Can you find a nonzero \( D \) so that \( C \otimes D = (0, 0, 0, 0) \)?

Solution.

(a) Here \( n = 4 \). The transform \( Fc \) is \( 5(1, 1, 1, 1) - (4, 0, 0, 0) = (1, 5, 5, 5) \). The right side has transform \( (1, 1, 1, 1) \). Multiplication (or division!) gives \( (1, 2, .2, .2) = .8(1, 0, 0, 0) + .2(1, 1, 1, 1) \) which comes from \( .8(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) + .2(1, 0, 0, 0) = (1, 2, .2, .2) = d \).

(b) The transform \( FC \) is \( 4(1, 1, 1, 1) - (4, 0, 0, 0) = (0, 4, 4, 4) \). We can’t divide by zero! The vector \( D = (1, 1, 1, 1) \) solves \( C \ast D = (0, 0, 0, 0) \).

Note for the future. Express the same problem with circulant matrices:

\[
\begin{bmatrix}
  4 & -1 & -1 & -1 \\
  -1 & 4 & -1 & -1 \\
  -1 & -1 & 4 & -1 \\
  -1 & -1 & -1 & 4 \\
\end{bmatrix}
\begin{bmatrix}
  .4 & .2 & .2 \\
  .2 & .4 & .2 \\
  .2 & .2 & .4 \\
  .2 & .2 & .2 \\
\end{bmatrix}
= I
\]
(b) No solution when the matrix is singular: a zero eigenvalue! (The eigenvalues are the discrete transforms!!)

\[
\begin{bmatrix}
3 & -1 & -1 & -1 \\
-1 & 3 & -1 & -1 \\
-1 & -1 & 3 & -1 \\
-1 & -1 & -1 & 3 \\
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]
2) (36 pts.)

(a) If \( f(x) = e^{-x} \) for \( 0 \leq x \leq 2\pi \), extended periodically, find its (complex)
Fourier coefficients \( c_k \).

(b) What is the decay rate of those \( c_k \) and how could you see the decay rate from the function \( f(x) \)?

(c) Compute \( \sum_{-\infty}^{\infty} |c_k|^2 \) for those \( c \)'s as an ordinary number. [1 point question: How in the world could you find \( \sum_{-\infty}^{\infty} |c_k|^4 \)? Don’t try!]

(d) Solve this periodic differential equation to find \( u(x) \):

\[
 u''(x) + u(x) = \delta(x) + \text{periodic train of deltas}
\]

**Solution.**

(a) \[ c_k = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-x} e^{-ikx} \, dx = \frac{e^{-(1+ik)x}}{-2\pi(1+ik)} \bigg|_{0}^{2\pi} = \frac{1 - e^{-(1+ik)2\pi}}{2\pi(1+ik)} = \frac{1 - e^{-2\pi}}{2\pi(1+ik)} \]

(b) Decay rate \( \frac{1}{k} \) because \( f(x) \) jumps from \( e^{-2\pi} \) to 1 at the end of every \( 2\pi \) period.

(c) \[ \sum_{-\infty}^{\infty} |c_k|^2 = \frac{1}{2\pi} \int_{0}^{2\pi} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-2x} \, dx = \frac{1 - e^{-4\pi}}{4\pi} \]

To find \( \sum |c_k|^4 \) we want the function \( F(x) \) whose Fourier coefficients are \( c_k^2 \). By the convolution rule \( F(x) \approx f \ast f \) (which is painfully computable since \( e^{-x} \) is easy to integrate).

(d) \( (ik + 1)c_k = \frac{1}{2\pi} \) so \( c_k = \frac{1}{2\pi(1+ik)} \), which agrees with part (a) after dividing by the constant: \( u(x) = \frac{f(x)}{1 - e^{-2\pi}} = \sum \frac{e^{ikx}}{2\pi(1+ik)} \).
3) (34 pts.) Suppose $f(x)$ is a half-hat function ($-\infty < x < \infty$).

$$f(x) = \begin{cases} 
1 - x & \text{for } 0 \leq x \leq 1 \\
0 & \text{for all other } x
\end{cases}$$

(a) Draw a graph of $f(x)$ on the whole line $-\infty < x < \infty$ and ALSO a graph of its derivative $g(x) = df/dx$.

(b) What is the transform (Fourier integral) $\hat{g}(k)$ of $df/dx$?

(c) What is the transform $\hat{f}(k)$ of $f(x)$? Does it have the decay rate you expect? What is $\hat{f}(0)$?

(d) Christmas present: Is the convolution $f(x) * f(x)$ of the half-hat with itself equal to the usual full hat $H(x)$? (Yes or no answer, 4 points).

THANK YOU FOR TAKING 18.085! 18.086 will be good small projects in scientific computing.

Solution.

(a) $g(x) = \delta(x)$—unit square wave on $[0, 1]$

(b) $\hat{g}(x) = 1 - \frac{e^{-ik}}{ik} = \frac{ik - 1 + e^{-ik}}{(ik)^2}$

(c) $\hat{f}(k) = \frac{\hat{g}(k)}{ik} = \frac{ik - 1 + e^{-ik}}{(ik)^2} = \frac{ik - 1 + (1 - ik - k^2/2 \cdots)}{(ik)^2}$

$$= \left( \frac{1}{2} \text{ at } k = 0 \right) = \text{area under } f(x)!$$

Decay rate $\frac{1}{k}$ because $f(x)$ has a step at $x = 0$.

(d) No way.