**18.085/18.0851 Comp. Science. Engineering I** **Name:** ______________
**Summer 2020**
**Final Examination**
**8/10/2020**
**Time Limit: Unlimited** ______________

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This exam contains 8 pages (including this cover page) and 7 questions plus 1 bonus questions.

Total of points is 63.

You are allowed all your written notes for the course. There will be no use of electronic devices except for looking up specific MATLAB commands online

Grade Table (for instructor use only)

| Question | Points | Score |
|---|---|---|
| 1 | 4 | |
| 2 | 4 | |
| 3 | 5 | |
| 4 | 5 | |
| 5 | 10 | |
| 6 | 25 | |
| 7 | 10 | |
| Total: | 63 | |

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1. (4 points) Which of the following are the appropriate Fourier basis functions over the interval $(-L, L)$? Here $n$ is an integer. Select all that applies

   A. $\sin(nx/L), \cos(nx/L)$

   B. $\sin(n\pi x/L), \cos(n\pi x/L)$

   C. $e^{\pm inx/L}$

   D. $e^{\pm in\pi x/L}$

2. (4 points) Which of the following are suitable Ansatz to the damped harmonic oscillator,

$$m\frac{d^2u}{dt^2} + \nu\frac{du}{dt} + cu = 0 \tag{1}$$

   where $u$ describes the displacement from equilibrium a ball of mass $m$ connected to a spring with Hooke's constant $c$ and experiencing a friction with coefficient $\nu$. Circle all that applies

A. $u(t) = \alpha \cos(bt) + \beta \sin(bt)$, with $\alpha, \beta, b \in \mathbb{R}$ to be solved

B. $u(t) = A \exp(bt)$, with $A, b \in \mathbb{C}$ to be solved

C. $u(t) = A \exp(ibt^2)$, with $A, b \in \mathbb{C}$ to be solved

D. $u(t) = A \exp(ibt)$, with $A, b \in \mathbb{C}$ to be solved

3. (5 points) Circle all true statements

   A. The solution to the heat equation in $\mathbb{R}$ can be discontinuous, depending on the initial condition.

   B. The non-smoothness of the boundary condition of the Laplace's equation can chip away the convergence rate of the finite difference approximation

   C. In an under-damped system, the damping effect is dominated by the inertia and/or the stiffness of the spring

   D. There should not be any terms involving $r^m$ for $m > 0$ in the solution to the **interior** domain of the Laplace's equation. Here $r$ is the radial coordinate in the polar system.

   E. The finite element method via mesh generation is less flexible in adapting a complex geometry of the domain compared with the finite difference method.

4. (5 points) Suppose I perform the FFT on a 128 data points, originally generated by the signal $\sin(2\pi * 3t)$, where $t \in [0.5362, 3.5345]$. Then when I plot $k$ vs. the magnitudes of discrete Fourier transform coefficients, $\hat{u}_k^d$, for $k = 0, ...127$, for what values of $k$ should I anticipate the plot to peak? Circle all that applies

   A. $k = 0$

   B. $k = 3$

   C. $k = 124$

   D. $k = 125$

   E. $k = 127$.

| $f(t)$ | $\mathscr{L}\{f(t)\} = F(s)$ | *Region of convergence* |
|---|---|---|
| $c$, $c$ a constant | $\frac{c}{s}$ | $\mathrm{Re}(s) > 0$ |
| $t$ | $\frac{1}{s^2}$ | $\mathrm{Re}(s) > 0$ |
| $t^n$, $n$ a positive integer | $\frac{n!}{s^{n+1}}$ | $\mathrm{Re}(s) > 0$ |
| $e^{kt}$, $k$ a constant | $\frac{1}{s-k}$ | $\mathrm{Re}(s) > \mathrm{Re}(k)$ |
| $\sin at$, $a$ a real constant | $\frac{a}{s^2+a^2}$ | $\mathrm{Re}(s) > 0$ |
| $\cos at$, $a$ a real constant | $\frac{s}{s^2+a^2}$ | $\mathrm{Re}(s) > 0$ |
| $e^{-kt}\sin at$, $k$ and $a$ real constants | $\frac{a}{(s+k)^2+a^2}$ | $\mathrm{Re}(s) > -k$ |
| $e^{-kt}\cos at$, $k$ and $a$ real constants | $\frac{s+k}{(s+k)^2+a^2}$ | $\mathrm{Re}(s) > -k$ |

5. (10 points) Solve the following differential equation using Laplace transform

$$\begin{cases} \frac{d^2u}{dt^2} - \frac{du}{dt} = 0 \\ u(0) = 0 \\ u'(0) = 1 \end{cases} \tag{2}$$

6. (25 points) Suppose we have the Laplace's equation $\Delta u = 0$ INSIDE a square defined by $x = 0$, $x = 1$, $y = 0$, and $y = 1$, subject to the following boundary condition

$$\begin{cases} u(x,0) = 0 \\ u(x,1) = 0 \\ u(0,y) = 0 \\ u(1,y) = \sin(n\pi y) \end{cases} \tag{3}$$

where $n$ is a fixed integer.

a)(10 points) Solve the Laplace equation analytically.
Two hints:

- Hint 1: Recall the orthogonality of $\sin(l\pi x)$ for $l = 1, 2, 3, ....$ By leveraging the orthogonality, you should not have to do any Fourier integrals explicitly
- Hint 2: Be very careful with dummy and non-dummy indices. Here $n$ is a non-dummy index.

b)(10 points) We set up the rectangular grid by dividing up the $x-axis$ and $y-axis$ into $N-by-N$ grids and run the finite difference scheme to obtain the numerical solution at each inner node $(x_i, y_j)$, where $i = 1, ..., N$ and $j = 1, ..., N$. Note that $x_0 = 0$, $x_{N+1} = 1$, $y_0 = 0$ and $y_{N+1} = 1$. Let $l$ be the linearized index (aka. global index in my notes) of all inner nodes. Note that $l = 1, 2, ..., N^2$.

Suppose the MATLAB function "$laplace1(N)$" implements the second-order finite difference scheme on the $N-by-N$ grid and spits out the numerical solution at each inner node $(l = 1, ..., N^2)$ . On the other hand, suppose the MATLAB function $laplace2(N)$" implements the formula in part a) and spits out the analytic solution at the same inner nodes.

Now we want to evaluate the order of convergence, measured by the $L^2-$ error, as $N$ gets large. Below you can find some incomplete code. **You are welcome to look up MATLAB commands online, but please do not open any codes on Stellar**:

i) Fill up the missing line (denoted as "......")
ii) Describe the output of the code. Here is the instruction for your answer:

- Is the output a number, an array, a plot, or something else?
- If it is a number or array, explicitly write it out or describe what each entry is.
- If it is a plot, indicate key features of the plot ( x-axis, y-axis, slope (if it's a line), period (if it's a sine function), etc).

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N_max = 100;

error_L_2 = zeros(N_max, 1);

for N=1:N_max
    u = laplace1(N);
    v = laplace2(N);
    error_val = 0;

    for n=1:N^2
        error_val = ......
    end

    error_val = sqrt(1/N * error_val);
    error_L_2(N) = error_val;
end

N_array = 1:1:N_max;
loglog(N_array, error_L_2)
```

c) (5 points) Suppose now we try to use the same machinery in part a) and the code in part b) for the problem with the following boundary condition:

$$\begin{cases} u(x,0) = 1 \\ u(x,1) = 1 \\ u(0,y) = 1 \\ u(1,y) = \sin(n\pi y) \end{cases} \tag{4}$$

How and why would the output of the code in part b) change? Here's the instruction for your answer:

- If it is a number or array, explicitly write them out or describe how each entry changes.
- If it is a plot, indicate how certain key features of the plot ( x-axis, y-axis, slope (if it's a line), period (if it's a sine function), etc).

7. (10 points) Compute the Fourier transform of the following function (factor of $2\pi$ would be forgiven)

$$f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x \geq 0 \end{cases} \tag{5}$$

(Bonus) Please summarize your understandings of the research talks on Wednesday, August 12th, 2020. A reasonable summary of each research talk will be worth 2 points, and up to 8 points will be granted. This part is optional and is due by 11:59pm of August 12, 2020.