True-False review questions on sequences and series

Each of the statements below is either true or false. If true, prove. If false, give a counterexample. \( \langle a_n \rangle \) is a sequence, another notation \( \sum_{n=1}^{\infty} a_n \).

1. Given \( \epsilon > 0 \), \( a_n \geq 1 - \epsilon \) for \( n \geq 1 \) \( \Rightarrow \lim_{n \to \infty} a_n = 1 \).

2. \( \langle x_n \rangle \) converges \( \Rightarrow \langle x_n \rangle \) is a Cauchy sequence.

3. a) \( x_n \to 0 \) \( \iff \left| x_n \right| \to 0 \)
   b) \( x_n \to L \) \( \iff \left| x_n \right| \to |L| \)

4. \( \langle a_n \rangle \) positive, decreasing \( \Rightarrow \sum (-1)^n a_n \) converges.

5. \( \langle a_n \rangle \) diverges \( \Rightarrow a_n \to \infty \) or \( a_n \to -\infty \).

6a. \( \langle x_n \rangle \) converges \( \Rightarrow \langle x_n \rangle \) bounded for all \( n \).
   b. \( n a_n \to 0 \) \( \Rightarrow \sum a_n \) converges.

7. For all \( \epsilon > 0 \), \( |a_{n+1} - a_n| < \epsilon \) for \( n \geq 1 \) \( \Rightarrow \langle a_n \rangle \) converges.

8a. \( \langle x_n \rangle \to 0 \) \( \iff \langle \frac{1}{x_n} \rangle \to \infty \).

10. \( x = \sup \langle a_n \rangle \) \( \Rightarrow x \) is a cluster pt. of \( \langle a_n \rangle \).

11. If \( \left| \frac{a_{n+1}}{a_n} \right| < 1 \) for all \( n \), then \( \sum a_n \) converges.

12a. \( \sum a_n \) converges \( \Rightarrow a_n \to 0 \).
   b. \( \sum a_n \) converges \( \iff \sum \frac{1}{a_n} \).

13. Every power series has a positive radius of convergence. \( R \), or \( R = \infty \).

14. \( a_n \to L \), \( L > 0 \) \( \Rightarrow a_n > 0 \) for \( n \geq 1 \).

15. \( a_n \to L \), \( L \geq 0 \) \( \Rightarrow a_n \geq 0 \) for \( n \geq 1 \).
1. Given \( c > 0 \),
\[
\left| \frac{a_n}{n^2} - \frac{1}{n^2} \right| = \left| \frac{1 - c}{n^2} \right| < \frac{1}{n^2 + 1} + \frac{1}{n^2 + 2} \tag{by \( \Delta \neq \)}
\]
\[
< \frac{1}{n^2} + \frac{2}{n^2} = \frac{3}{n^2};
\]
\[
\frac{4}{n^2} < c \text{ if } n > \frac{4}{c}.
\]

2. By the ratio test,
\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)!}{n!} \frac{c^n}{c^{n+1}} \frac{(n+2)!}{(n+1)!} = \frac{(n+1)(n+2)}{n+2} = \frac{n+2}{n+1} \frac{c^n}{c^{n+1}} \frac{n+2}{n+1} \frac{c^n}{c^{n+1}}
\]
\[
\rightarrow 4|x| < 1 \Leftrightarrow |x| < \frac{1}{4}.
\]
\( a_n \) converges, \( \Rightarrow |x| < \frac{1}{4} \); diverges, \( \Rightarrow |x| > \frac{1}{4} \).

3. \( a_n = \frac{c^n}{n^n} \) if \( c > 1 \).

4. \( a_n = \frac{1}{n^2} \) is decreasing for \( n > 1 \): \[
\frac{a_{n+1}}{a_n} = \frac{n+1}{n} \frac{c^n}{c^{n+1}} \frac{n+1}{n} = \frac{c}{n+1} \frac{c^n}{c^{n+1}} \frac{n+1}{n+1} \frac{c^n}{c^{n+1}} \frac{n+1}{n+1} \frac{c^n}{c^{n+1}} < 1 \text{ if } n > c - 1.
\]

5. \( a_n \) converges, \( \rightarrow |x| < \frac{1}{4} \); diverges, \( \Rightarrow |x| > \frac{1}{4} \).

6. Let \( p_i = i^{th} \) prime, and \( n_i = p_i^k \) (for a fixed \( k \))

Then \( \frac{h(n_i)}{s(n_i)} = \frac{p_i}{k} \), \( s(n_i) \) converges to \( \frac{1}{k} \), and \( \frac{1}{k} \) is a cluster point of \( s(n_i) \). By the cluster pt. thm \( S(n) \) doesn't exist, for if it had the limit \( L \), all subsequences would have \( \lim_{n \to 0} a_n = L \).

7. Choose \( x_n \) to be any element of \( S \)

such that \( x_n > n \).

Such an elt. exist since \( S \) is nonempty and not bounded above (if there were no such \( x_n \), then \( n \) would be an upper bound for \( S \)).

Then \( \lim_{n \to 0} x_n = \infty \), since \( \frac{1}{n} \) diverges, given \( M > 0 \), \( x_n > n > M \) for all \( n > M \).

8. You can use the vertical test to prove that \( \frac{c^n}{s^n} \) converges, and therefore \( \lim_{n \to 0} \frac{c^n}{s^n} = 0 \), by the \( n^{th} \) term test for divergence.