Practice Quiz 2
18.100B R2 Fall 2010

Closed book, no calculators.

YOUR NAME: ________________________________

This is a 30 minute in-class exam. No notes, books, or calculators are permitted. Point values are indicated for each problem. Do all the work on these pages.
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**Problem 1.** [5+5+5 points]

Let \((X, d)\) be a metric space.

(a) State the definition of a connected subset of \(X\) via separated sets, as in Rudin.

(b) Let \((X, d)\) be connected (i.e. \(X\) is connected as a subset of \((X, d))\). Show that a subset \(A \subset X\) is both open and closed if and only if \(A = \emptyset\) or \(A = X\). (This was a homework problem, but the task is to reprove this fact.)
(c) Suppose that \((X, d)\) is a metric space with the following property: A subset \(A \subset X\) is both open and closed if and only if \(A = \emptyset\) or \(A = X\). Then show that \((X, d)\) is connected (i.e. \(X\) is connected as a subset of \((X, d)\)).
Problem 2. [10+10 points]

(a) Find $\liminf_{n \to \infty}$ and $\limsup_{n \to \infty}$ for each of the following sequences. Are these sequences bounded and/or convergent?

$$a_n = \sin \left( \frac{n\pi}{4} \right), \quad b_n = \frac{(-1)^n}{n^{3/2}}.$$
(b) Let \((a_n), (b_n)\) and \((c_n)\) be sequences in \(\mathbb{R}\) such that for all \(n \geq N\) we have \(a_n \leq b_n \leq c_n\). Assume also that \(\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L\) for some real number \(L\).

Prove that \(\lim_{n \to \infty} b_n = L\).
Problem 3. [10 points] Assume that $\sum_{n=1}^{\infty} a_n$ is a convergent series and that $a_n \geq 0$ for all $n \geq N$. Prove that $\sum_{n=1}^{\infty} \frac{1}{n} \sqrt{|a_n|}$ converges. (Hint: You can use the general inequality $2xy \leq x^2 + y^2$ for $x, y \in \mathbb{R}$.)
Problem 4. [20 points: +4 for each correct, −4 for each incorrect; no proofs required.]
(Hint: Note the penalty – it may be wise to leave some questions unanswered.)

a) Let \((X, d)\) be a metric space, and let \(E \subset X\). Then the closure of \(E\) is equal to the set \(L(E)\) of all limits of sequences in \(E\):

\[
L(E) = \{x \in X \mid \exists (x_n)_{n \in \mathbb{N}} \subset E : \lim_{n \to \infty} x_n = x\}.
\]

TRUE    FALSE

b) If \(\sum_{n=1}^{\infty} a_n\) is convergent and \(a_n \geq 0\) then \(a_n \to 0\).

TRUE    FALSE

c) The subset \(\{z \in \mathbb{Q} \mid |z| < 1\}\) of \(\mathbb{Q}\) is connected.

TRUE    FALSE

d) Let \((x_n)\) be a sequence in the metric space \((X, d)\) such that \(d(x_n, x_{n+1}) \leq \frac{1}{n}\). Then \((x_n)\) is a Cauchy sequence.

TRUE    FALSE

e) Suppose \(\sum_{n=1}^{\infty} c_n z^n\) is a power series with convergence radius \(R = 2\) and such that it converges for \(z = 2\). Then it converges for all other \(z \in \mathbb{C}\) with \(|z| = 2\).

TRUE    FALSE