Problems:

1) (10 pts) Prove that the empty set is a subset of every set.

2) (10 pts) If \( x, y \) are complex, prove that
\[
| |x| - |y| | \leq |x - y|.
\]
(Hint: This is equivalent to proving the following two inequalities: \(|x| \leq |x - y| + |y| \) and \(|y| \leq |x - y| + |x| \). Why?)

3) (10 pts) Find \( \text{sup} \ M \) and \( \text{inf} \ M \) for:
   a) \( M = \{ \frac{|x|}{1 + |x|} : x \in \mathbb{R} \} \),
   b) \( M = \{ \frac{x}{1 + x} : x > -1 \} \),
   c) \( M = \{ x + \frac{1}{x} : \frac{1}{2} < x < 2 \} \).

4) (10 pts) Let:
   a) \( S \) be the set of all natural numbers that are not divisible by a square number;
   b) \( T \) be the set of all natural numbers that have exactly three prime divisors;
   c) \( U \) be the set of all natural numbers that are less or equal than 200.
Determine \( S \cap T \cap U \) explicitly.

5) (10 pts) Let \( X \) and \( Y \) be two disjoint sets. Suppose further that \( X \sim \mathbb{R} \) and that \( Y \sim \mathbb{N} \)
   (i.e. the set \( Y \) is countable). Show that \( Z = X \cup Y \) satisfies \( Z \sim \mathbb{R} \).

6) (10 pts) Construct a bounded set of real numbers with exactly three limit points. In addition,
   construct a bounded set of real numbers with countably many limit points.

7) (10 pts) Let \( E \) be a subset of a metric space. The \textit{interior} \( E^o \) is defined by
\[
E^o = \{ x \in E : x \text{ is an interior point} \}.
\]
   a) Prove that \( E^o \) is always open.
   b) Prove that \( E \) is open if and only if \( E^o = E \).
   c) If \( G \subseteq E \) and \( G \) is open, prove that \( G \subseteq E^o \).
Extra problems:

1) Consider the function \( f : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\} \) with \( f(z) = 1/z \). Sketch the following sets in the complex plane:
   a) \( f(\mathbb{R} \setminus \{0\}) \),
   b) \( f(B_r) \) where \( B_r = \{ z \in \mathbb{C} : |z| = r \} \) and \( r > 0 \),
   c) \( f(i\mathbb{R} \setminus \{0\}) \),
   d) \( f(A) \) where \( A = \{ z \in \mathbb{C} : \text{Re} \ z = 1 \} \).
[Recall that, for a given function \( f : X \rightarrow Y \), the set \( f(E) = \{ f(x) : x \in E \} \) is the image of a subset \( E \subseteq X \) under \( f \).]

2) A complex number \( z \) is said to be 
   algebraic if there are integers \( a_0, \ldots, a_n \), not all zero, such that
   \( a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0 \).
   Prove that the set of all algebraic numbers is countable. 
   \textit{Hint:} For every positive integer \( N \) there are only finitely many equations with
   \( n + |a_0| + |a_1| + \cdots + |a_n| = N \).

3) If you think of the existence of a 1-1 map from \( A \) into \( B \) as saying that \( A \) is ‘not bigger than’ \( B \) (think \( \leq \)). Then this exercise proves that: if \( A \) is not bigger than \( B \) and \( B \) is not bigger than \( A \), then \( A \) and \( B \) are the same size.
   Prove the Schroeder-Bernstein theorem
   If \( A \) and \( B \) are any two sets, \( f \) is a 1-1 map from \( A \) into \( B \) and \( g \) is a 1-1 map from \( B \) into \( A \), then there exists a map \( F \) from \( A \) to \( B \) which is 1-1 and onto, i.e., \( A \sim B \).
   by the following steps (due to Birkhoff and MacLane):
   i) Define ‘ancestors’ as follows: Let \( a \in A \), if \( a \in g(B) \) then we call \( g^{-1}(a) \) the first ancestor of \( a \) (we call \( a \) itself the zero\(^{th} \) ancestor of \( a \)). If \( g^{-1}(a) \) is in \( f(A) \) then we call \( f^{-1}(g^{-1}(a)) \) the second ancestor of \( a \). If this is in the image of \( g \), then we call \( g^{-1}(f^{-1}(g^{-1}(a))) \) the third ancestor of \( a \) and so on. 
   Show that this divides \( A \) into three disjoint subsets: \( A_\infty \) made up of the elements that have infinitely many ancestors, \( A_e \) made up of the elements that have an even number of ancestors, and \( A_o \) made up of the elements that have an odd number of ancestors.
   ii) Show that you can partition \( B \) into three similar subsets: \( B_\infty \), \( B_e \), and \( B_o \).
   iii) Identify \( f(A_\infty) \), \( f(A_e) \), and \( f(A_o) \).
   iv) Define
   \[
   F(a) = \begin{cases} 
   f(a) & \text{if } a \in A_\infty \cap A_e \\
   g^{-1}(a) & \text{if } a \in A_o
   \end{cases}
   \]
   and show that \( F \) is a 1-1 correspondence between \( A \) and \( B \).

4) Show that if \( Sq = [0,1] \times [0,1] \) is the unit square and \( I = [0,1] \) is one of its sides, then \( Sq \sim I \).