Problem 1.

Proof. It is true that for any two sets $A$, $B$, the intersection $A \cap B$ is a subset of $A$. Now consider $\phi = A \cap A^c$. So $\phi$ is a subset of $A$ for any set $A$. \hfill \Box

Problem 2.

Proof. Notice that $||x| - |y|| \leq |x - y|$ ⇔ $|x| - |y| \leq |x - y|$ and $|y| - |x| \leq |x - y|$.
So we only need to prove that $|x| \leq |x - y| + |y|$ and $|y| \leq |x - y| + |x|$.
But both of them is a consequence from the triangle inequality $|a - b| \leq |a - c| + |b - c|$. \hfill \Box

Problem 3.

(a) $M = \{ \frac{|x|}{1 + |x|} : x \in \mathbb{R} \}$.

Proof. Notice that

\[
\frac{|x|}{1 + |x|} = \frac{1}{\frac{1}{|x|} + 1}
\]

so if $|x| < |y|$ then

\[
\frac{|x|}{1 + |x|} < \frac{|y|}{1 + |y|}.
\]

Thus the supremum is $\frac{1}{0+1} = 1$ and the infimum is $\frac{0}{1+0} = 0$. \hfill \Box

(b) $M = \{ \frac{x}{1 + x} : x > -1 \}$.

Proof. We can change the variable $x$ to $y$, $\frac{x}{1 + x} = \frac{y - 1}{y} = 1 - \frac{1}{y}$, where $y = x + 1$. From $x > -1$, we have $y > 0$. Notice that

$\frac{1}{y}$ decreases $\Rightarrow$ $y$ increases $\Rightarrow$ $(1 - \frac{1}{y})$ increases,

so the supremum is $1 - 0 = 1$ and the infimum is $-\infty$ (because for every $N > 1$ we have

\[
\frac{\frac{N}{1-N}}{1 + \frac{N}{1-N}} = -N
\]

and so the infimum is less than $-N$). \hfill \Box
(c) $M = \{x + \frac{1}{x} | 1/2 < x < 2\}$.

Proof. It is always true that

$$\frac{a + b}{2} \geq \sqrt{ab},$$

for instance, if square both sides and rearrange, this is the same as saying $a^2 + b^2 \geq 0$.

Thus, we see that

$$x + \frac{1}{x} \geq 2\sqrt{x \cdot \frac{1}{x}} = 2$$

Since setting $x = 1$ in $x + \frac{1}{x}$ we get 2, we know that $\inf M = 2$.

Suppose we have $x_1 > x_2$, consider

$$x_1 + \frac{1}{x_1} - \left( x_2 + \frac{1}{x_2} \right) = \frac{(x_1 - x_2)(x_1x_2 - 1)}{x_1x_2} > 0,$$

i.e. $x + \frac{1}{x}$ is an increasing function; if $x_1, x_2 < 1$, then

$$x_1 + \frac{1}{x_1} - \left( x_2 + \frac{1}{x_2} \right) = \frac{(x_1 - x_2)(x_1x_2 - 1)}{x_1x_2} < 0,$$

i.e. $x + \frac{1}{x}$ is a decreasing function. Then the sup must be obtained at the boundary of $(1/2, 2)$.

Since

$$\lim_{x \to 2} \left( x + \frac{1}{x} \right) = \lim_{x \to 1/2} \left( x + \frac{1}{x} \right) = \frac{5}{2},$$

we have $\sup M = \frac{5}{2}$.

□

Problem 4.

Proof. The answer is:

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□
Problem 5.

Proof. From \( X \sim \mathbb{R} \), then there is a \( 1 - 1 \) mapping \( \alpha : X \to \mathbb{R} \). Similarly we have a \( 1 - 1 \) mapping \( \beta : Y \to \mathbb{N} \). So to prove \( Z = X \cup Y \sim \mathbb{R} \), we only need to prove that there is a \( 1 - 1 \) mapping \( \gamma : Z \to \mathbb{R} \). It is equivalent to show that there is \( 1 - 1 \) mapping \( \delta : \mathbb{N} \cup \mathbb{R} \to \mathbb{R} \). The \( \delta \) can be constructed by the following method:

\[
\delta(x) = \begin{cases} 
  x & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}; \\
  x & \text{if } x \in \mathbb{Z} \subset \mathbb{R} \text{ and } x \leq 0; \\
  2x & \text{if } x \in \mathbb{Z} \subset \mathbb{R} \text{ and } x > 0; \\
  2x + 1 & \text{if } x \in \mathbb{N}.
\end{cases}
\]

It is easy to check that it is an \( 1 - 1 \) mapping. \( \square \)

Problem 6.

Proof. Consider the sets

\[
A_0 = \{ \frac{1}{n} | n \in \mathbb{N} \}, \\
A_1 = \{ \frac{1}{n} + 1 | n \in \mathbb{N} \}, \\
A_2 = \{ \frac{1}{n} + 2 | n \in \mathbb{N} \}.
\]

Then \( A_i \) has only one limit point \( i \), for \( i = 0, 1, 2 \). If let \( A = \bigcup_{i=0}^{2} A_i \), we get a bounded set \( A \) with three limit points.

Consider the set

\[
\mathcal{A} = \{ \frac{1}{n} + \frac{1}{m} : n, m \in \mathbb{N} \},
\]

we check that the limit points of \( \mathcal{A} \) are precisely the points in \( A_0 \cup \{0\} \). Indeed, if we fix \( n_0 \in \mathbb{N} \) then the set

\[
\{ \frac{1}{n_0} + \frac{1}{m} : m \in \mathbb{N} \},
\]

has \( \frac{1}{n_0} \) as a limit point and is a subset of \( \mathcal{A} \), hence \( \mathcal{A} \) has \( \frac{1}{n_0} \) as a limit point, for any \( n_0 \in \mathbb{N} \). Also \( A_0 \subseteq \mathcal{A} \) so zero is a limit point of \( \mathcal{A} \). To see that there are no other limit points, pick a point \( x \in \mathbb{R} \) that is not equal to \( \frac{1}{n} \) for any \( n \in \mathbb{N} \), we show that \( x \) is not a limit point of \( \mathcal{A} \). We can find \( N \in \mathbb{N} \) such that

\[
\frac{1}{N} < x < \frac{1}{N - 1}
\]

Pick \( \varepsilon > 0 \) small enough so that

\[
\frac{1}{N} < x - \varepsilon < x < x + \varepsilon < \frac{1}{N - 1}
\]

and notice that there are at most finitely many elements of \( \mathcal{A} \) in \( (x - \varepsilon, x + \varepsilon) \). Here is one way to see this: if \( n \) and \( m \) are both bigger than \( 2N \) then \( \frac{1}{n} + \frac{1}{m} < \frac{1}{N} \), if \( n < N \) then \( \frac{1}{n} + \frac{1}{m} > \frac{1}{N} \), while if \( 2N \geq n > N \) then

\[
\frac{1}{N} < \frac{1}{n} + \frac{1}{m} \iff -\frac{1}{m} < \frac{1}{n} - \frac{1}{N} = \frac{N - n}{nN} \iff m < \frac{nN}{n - N},
\]

finally if \( n = N \), and \( m \) is large enough then \( \frac{1}{n} + \frac{1}{m} < x - \varepsilon \). So there are finitely many possible pairs \( (n, m) \) with \( x - \varepsilon < \frac{1}{n} + \frac{1}{m} < x + \varepsilon \).
Since there are only finitely many elements of $A$ inside $(x - \varepsilon, x + \varepsilon)$ we can find $k \in \mathbb{N}$ so that $(x - \frac{\varepsilon}{k}, x + \frac{\varepsilon}{k})$ contains no element of $A$ except possibly $x$ itself. This proves that $x$ is not a limit point of $A$.

Problem 7.

Proof. (a) The points in $E^0$ are interior points of $E$, to show that $E^0$ is open we need to show that they are interior points of $E^0$. Given $x \in E^0$, by definition, there exist a open ball $x \in B_r(x) \subset E$. Consider an open ball $B_{r/3}(x) \subset B_r(x)$. Then for any point $y \in B_{r/3}(x)$, $B_{r/3}(y) \subset B_r(x) \subset E$, so $y \in E^0$. Then $B_{r/3}(x)$ is an open ball in $E^0$. So $x$ is an interior point of $E^0$.

(b) If $E = E^0$, from (a) we know that $E$ is open. Conversely, if $E$ is open, all points in $E$ are interior points, so $E \subset E^0$. From $E^0 \subset E$ we have $E = E^0$.

(c) Since $G$ is open, so for any point $g \in G$, we have an open ball $B_r(g) \subset G \subset E$. So $g$ is also an interior point of $E$. Then $G \subset E^0$. 

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