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For bounded functions $f, g : [a, b] \to \mathbb{R}$, we use the notation $||f|| = \sup\{|f(x)| : x \in [a, b]\}$ and $d(f, g) = ||f - g||$.

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We have $f_n \to f$ and $g_n \to g$ uniformly. We wish to show that $f_n g_n \to fg$ uniformly as well. Let $\epsilon > 0$. $f$ and $g$ are bounded by assumption, so pick $K \in \mathbb{R}$ with $||f||, ||g|| < K$. We may assume $K > \epsilon$. Pick a $\delta > 0$ with $\delta < \epsilon/(3K)$, and pick $N \in \mathbb{N}$ sufficiently large that for $n > N$, $||f - f_n||, ||g - g_n|| < \delta$, which is possible by Rudin 7.9. Note that for any such $n > N$, we have by Rudin 7.14

$$||f_n|| \leq ||f - f|| + ||f|| < \delta + K < 2K$$

And similarly $||g_n|| < 2K$. Let $x \in [a, b]$. We then have, for $n > N$,

$$|f(x)g(x) - f_n(x)g_n(x)| = |(f(x)g(x) - f(x)g_n(x)) + (f(x)g_n(x) - f_n(x)g_n(x))|$$

$$< |f(x)(g(x) - g_n(x))| + |g_n(x)(f(x) - f_n(x))|$$

$$= |f(x)| \cdot |g(x) - g_n(x)| + |g_n(x)| \cdot |f(x) - f_n(x)|$$

$$< K\delta + 2K\delta < \epsilon$$

Since this was true for any $x \in [a, b]$, we must have $||fg - f_n g_n|| < \epsilon$ for any $n > N$, which proves the result.
Let $F$ be the set of all continuous functions $f : [0, 1] \to \mathbb{R}$ with $f(0) = 0$ and $f(1) = 1$. We have to show that if $f \in F$, then $\hat{f} \in F$. We have

$$\hat{f}(0) = \frac{1}{4} f(2 \cdot 0) = \frac{1}{4} f(0) = 0$$

and

$$\hat{f}(1) = \frac{3}{4} f(2 - 1) + \frac{1}{4} = \frac{3}{4} + \frac{1}{4} = 1$$

We also need to show that $\hat{f}$ is continuous. At points $x \neq 1/2$ $\hat{f}$ is continuous by Rudin 4.7, we just need to show that it is continuous at $1/2$. Note that $\hat{f}(1/2) = 3/4(f(0)) + 1/4 = 1/4$

Let $\epsilon > 0$. Pick $\delta > 0$ such that

$$|x - 0| < 2\delta \implies |f(x) - f(0)| = |f(x)| < \epsilon$$

and

$$|x - 1| < 2\delta \implies |f(x) - f(1)| = |f(x) - 1| < \epsilon$$

Now suppose $|x - 1/2| < \delta$. We wish to show that $|\hat{f}(x) - \hat{f}(1/2)| = |\hat{f}(x) - 1/4| < \epsilon$. There are two possibilities.

If $x < 1/2$, then $|1 - 2x| < 2\delta$, and so $|f(2x) - 1| < \epsilon$. But then

$$|\hat{f}(x) - \hat{f}(1/2)| = \frac{1}{4} |f(2x) - 1| = \frac{1}{4} |f(2x) - 1| < \frac{\epsilon}{4}$$

Similarly, if $x > 1/2$, then again $|2x - 1| < 2\delta$, and so $|f(2x - 1)| < \epsilon$. Then

$$|\hat{f}(x) - \hat{f}(1/2)| = \frac{3}{4} |f(2x - 1) + \frac{1}{4} - \frac{1}{4}| = \frac{3}{4} |f(2x - 1)| < \frac{3\epsilon}{4}$$

In either case $|\hat{f}(x) - \hat{f}(1/2)| < \epsilon$, so $\hat{f}$ is continuous at $1/2$.

Suppose $f, g \in F$, and let $x \in [0, 1]$. If $x < 1/2$, we have

$$|\hat{f}(x) - \hat{g}(x)| = \frac{1}{4} |f(2x) - g(2x)| = \frac{1}{4} |f(2x) - g(2x)| \leq \frac{1}{4} \|f - g\| = \frac{1}{4} d(f, g)$$

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Similarly, if \( x \geq 1/2 \), we have

\[
|\hat{f}(x) - \hat{g}(x)| = |\left(\frac{3}{4} f(2x - 1) - \frac{1}{4}\right) - \left(\frac{3}{4} g(2x - 1) - \frac{1}{4}\right)|
\]

\[
= \frac{3}{4} |f(2x - 1) - g(2x - 1)| \leq \frac{3}{4} ||f - g|| = \frac{3}{4} d(f, g)
\]

In either case \( |\hat{f}(x) - \hat{g}(x)| \leq 3/4d(f, g) \), and so

\[
d(\hat{f}, \hat{g}) = ||\hat{f} - \hat{g}|| \leq \frac{3}{4} d(f, g)
\]

Now, \( \mathcal{F} \) is a metric space with metric \( d(\cdot, \cdot) \), and by what we have shown we can think of \( \mathcal{F} \to \mathcal{F} \) as a function which contracts distances by at least \( 3/4 \).

Now suppose that \( \mathcal{F} \) is actually a complete metric space. Then by the Contraction Mapping Theorem, Rudin 9.23 (which we have proved on a previous homework), there would have to be a unique element \( f \in \mathcal{F} \) with \( \hat{f} = f \). So we just have to prove that \( \mathcal{F} \) is complete.

Note that \( \mathcal{F} \subset \mathcal{C} = \mathcal{C}([0, 1], \mathbb{R}) \), the set of all continuous functions \([0, 1] \to \mathbb{R}\), and in fact the metric on \( \mathcal{F} \) is the restriction of the metric on \( \mathcal{C} \) By Rudin Theorem 7.15, \( \mathcal{C} \) is a complete metric space. But closed subsets of complete metric spaces are themselves complete (you should check this if it isn’t obvious to you), so if we can show that \( \mathcal{F} \subset \mathcal{C} \) is closed then we are done.

We will show that \( \mathcal{F} \subset \mathcal{C} \) is closed by showing that its complement is open. So let \( f \in \mathcal{C} \setminus \mathcal{F} \). Then either \( f(0) \neq 0 \) or \( f(1) \neq 1 \). Without loss of generality suppose \( f(0) \neq 0 \). Let \( \epsilon > 0 \) such that \( |f(0)| > 2\epsilon \). Then if \( d(f, g) < \epsilon \), in particular \( |f(0) - g(0)| < \epsilon \), and so \( |g(0)| > \epsilon \), and \( g \in \mathcal{F}^c \). In other words, \( B_\epsilon(f) \subset \mathcal{F}^c \), and so \( \mathcal{F}^c \) is open.

Another, more conceptual way to prove that \( \mathcal{F} \) is closed is to show that for any \( a \in [0, 1] \), the map \( ev_a : \mathcal{C} \to \mathbb{R} \) given by \( ev_a(f) = f(a) \) is continuous. But the inverse image of a closed set under a continuous map is closed, and so \( \mathcal{F} = ev_a^{-1}(0) \cap ev_b^{-1}(1) \) is also closed. Details are left to the interested reader.
For any $x \in [a, b]$, the sequence $f_1(x), f_2(x), \ldots$ is an alternating sequence of real numbers of decreasing norm, with the norm converging to 0. Hence by Rudin 3.43, or by a previous homework problem, the series $\sum_n f_n$ converges. Define a function (not necessarily continuous) $f : [a, b] \to \mathbb{R}$ by $f(x) := \sum_n f_n(x)$. Then the sequence of partial sums $s_n = \sum_{k=1}^n f_k$ converges pointwise to $f$. We wish to show that the convergence is uniform.

Let $\epsilon > 0$. We need to find an $N \in \mathbb{N}$ such for $n > N$ and any $x \in [a, b]$, $|f(x) - s_n(x)| < \epsilon$. We know that $f_k \to 0$ uniformly. So let $N \in \mathbb{N}$ be sufficiently large that $|f_k(x)| < \epsilon$ for all $k > N, x \in [a, b]$.

We now need the following

Lemma: Suppose $(a_n)$ is an alternating sequence as in Rudin 3.43, and $a = \sum_n a_n$. Then $|a| < |a_1|$.

Assume the Lemma for the moment. For any $n > N$, indeed any $n$, we have

$$f(x) - s_n(x) = \sum_{k=n+1}^{\infty} f_k(x)$$

But $\sum_{k=n+1}^{\infty} f_k(x)$ is itself an alternating series. By the Lemma, we then have

$$|f(x) - s_n(x)| = |\sum_{k=n+1}^{\infty} f_k(x)| < |f_{n+1}(x)| < \epsilon$$

Since the choice of $N$ did not depend on the point $x$, we have $s_n \to f$ uniformly.

Proof of Lemma: We first show that $a_1$ and $a$ have the same sign. We have

$$a = \sum_{n=1}^{\infty} a_n = (a_1 + a_2) + (a_3 + a_4) + \cdots = \sum_{n=1}^{\infty} (a_{2n-1} + a_{2n})$$

All terms $a_{2n-1} - a_{2n}$ in the sum on the right have the same sign as $a_1$, and so $a$ must also have the same sign as $a_1$. 

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Now assume $a_1 > 0$. Then $a > 0$ as well, and $a - a_1 = \sum_{n=2}^{\infty} a_n$. But by the previous paragraph, the latter sum has the same sign as $a_2$, which is negative. Hence $a - a_1 < 0$, and $0 < a < a_1$, so $|a| < |a_1|$. The result follows for $a_1 < 0$ by replacing $a_n$ with $-a_n$. 
