We write $d = d_{SNCF}$ for the French Railroad metric on $\mathbb{R}^2$. In this problem we will often use the easily checked fact that if $(x_n)$ is a Cauchy sequence in any metric space, then $x_n \to x$ if and only if $x_{n_k} \to x$ for some subsequence $(x_{n_k})$.

So let $(x_n) \subseteq \mathbb{R}^2$ be a Cauchy sequence with respect to $d$. We will show that $(x_n)$ is convergent, and hence that $d$ is a complete metric. If $x_n \to 0$ then obviously we are done, so assume $(x_n)$ does not converge to 0. Note that since 0 lies on every line through the origin by definition, we have that $d(x_n, 0) = |x_n|$.

I claim that there exists $\epsilon > 0$ such that $|x_n| > \epsilon$ for all $n \in \mathbb{N}$. Indeed, if not then there exists a subsequence $(x_{n_k})$ with $|x_{n_k}| \to 0$ as $k \to \infty$, which means that the subsequence $(x_{n_k})$ converges to 0 with respect to $d$. Then since $(x_n)$ is Cauchy, $x_n \to 0$, contradiction.

Now let $N \in \mathbb{N}$ be sufficiently large that $d(x_n, x_m) < \epsilon$ for $n, m > N$. Then $x_n$ and $x_m$ must lie on the same line. If they didn’t, then

$$\epsilon > d(x_n, x_m) = |x_n| + |x_m| > \epsilon + \epsilon$$

Contradiction.

In other words, there exists a line $l \subseteq \mathbb{R}^2$ with $x_n \in l$ for $n > N$. For any $n, m > N$, we then have $d(x_n, x_m) = |x_n - x_m|$. Thus, $(x_n)$ is a Cauchy sequence with respect to the standard Euclidean metric on $\mathbb{R}^2$, since $d$ will agree that metric for sufficiently large $n$. Since $|\cdot|$ is complete, there exists
$x \in \mathbb{R}^2$ with $\lim_{n \to \infty} |x - x_n| = 0$. Lines are closed subsets of $\mathbb{R}^2$ with respect to $| \cdot |$, so $x \in l$. Then $d(x_n, x) = |x_n - x|$. Thus $x_n \to x$ with respect to $d$, and so $d$ is complete.

2

Let $E = \mathbb{Q} \cap [0, 1]$. $E$ is an infinite subset of a countably infinite set, hence is countably infinite. In other words, there exists a bijective function $f : \mathbb{N} \to E$. Define the sequence $(x_n)$ via $x_n = f(n)$. Note that $\overline{E} = [0, 1]

Let $F$ be the set of all subsequential limits of $E$. I claim that $F = [0, 1]$. Suppose that $x \in F$. Take a subsequence $(x_{n_k})$ converging to $x$. Then every neighbourhood of $x$ contains all but finitely many $(x_{n_k})$, and in particular intersects $E$. So $x \in \overline{E} = [0, 1]$. Conversely, suppose $x \in [0, 1]$. We will construct a subsequence $x_{n_k} \to x$ inductively. Let $n_1 = 1$. Suppose we have defined $n_1, n_2, \ldots n_k$. Consider the subset $A_{k+1} \subset \mathbb{N}$, defined by

$$A_{k+1} = \{n \in \mathbb{N} : |x - f(n)| < \frac{1}{k}\}$$

Recall that $x_n = f(n)$. Since $x$ is a limit point of $E$, $B_{1/k+1}(x)$ contains infinitely many points of $E$; since $f$ is surjective, this implies that $A_{k+1}$ is infinite. Thus we can pick a $n_{k+1} \in A_{k+1}$ with $n_{k+1} > n_k$.

Thus we have constructed a subsequence $(x_{n_k})$ with $|x - x_{n_k}| < 1/k$, which means that $x_{n_k} \to x$, so $x \in F$. Thus $F = [0, 1]$ and we are done.

3

We have a continuous function $f : [0, 1] \times [0, 1] \to \mathbb{R}$. For a fixed $x \in [0, 1]$, consider the function $h_x : [0, 1] \to \mathbb{R}$ defined by $h_x(y) = f(x, y)$. Then $h_x$ is continuous; indeed, for any $y \in [0, 1]$ and $\epsilon > 0$, take a $\delta > 0$ that works for $f$ and $\epsilon$ at $(x, y)$. Since $h_x$ is a continuous function on a compact set, it attains a finite maximum; in other words, for some $y_0 \in [0, 1]$, we have $f(x, y_0) = h_x(y_0) \geq h_x(y) = f(x, y)$ for all $y \in [0, 1]$. Then

$$g(x) = \sup_{y \in [0, 1]} \{f(x, y)\} = f(x, y_0)$$
Is well defined. We need to show that $g : [0, 1] \to \mathbb{R}$ is continuous. Note that we have not only proved that $g$ is well defined, but have also shown that for any $x \in [0, 1]$, there exists $y \in [0, 1]$ with $g(x) = f(x, y)$.

Let $x \in [0, 1]$. We need to show that $\lim_{z \to x} g(z) = g(x)$. So suppose this is false. Then there exists $\varepsilon > 0$ and a sequence $(x_n)$ with $x_n \to x$ but $|g(x_n) - g(x)| > \varepsilon$.

For each $x_n$, pick $y_n \in [0, 1]$ such that $g(x_n) = f(x_n, y_n)$. Now consider the sequence $((x_n, y_n))_{n \in \mathbb{N}}$. This a sequence in the compact set $[0, 1] \times [0, 1]$, hence has a convergent subsequence. In other words there exists $(x', y') \in [0, 1] \times [0, 1]$ with $(x_{n_k}, y_{n_k}) \to (x', y')$. This implies that $x_{n_k} \to x'$, but since this a subsequence of a convergent sequence, it must also converge to $x$, and so $x' = x$.

$f$ is continuous, and so

$$f(x, y') = \lim_{k \to \infty} f(x_{n_k}, y_{n_k}) = \lim_{k \to \infty} g(x_{n_k})$$

Hence, we must have $|g(x) - f(x, y')| \geq \varepsilon$. Pick $y \in [0, 1]$ with $f(x, y) = g(x)$. Then $f(x, y) \geq f(x, y')$, by the definition of $g$, and so

$$f(x, y) - f(x, y') \geq \varepsilon$$

On the other hand, $f$ is uniformly continuous, since it is a continuous function on a compact set. Pick a $\delta > 0$ such that

$$d((z, w), (z', w')) < \delta \implies d(f(z, w), f(z', w')) < \varepsilon/3$$

and $k$ sufficiently large that $|x - x_{n_k}|, |y' - y_{n_k}| < \delta/\sqrt{2}$. Then $|f(x, y) - f(x_{n_k}, y)| < \varepsilon/3$, and so

$$f(x_{n_k}, y) > f(x, y) - \varepsilon/3$$

Similarly, $|f(x_{n_k}, y_{n_k}) - f(x, y')| < \varepsilon/3$, and so

$$f(x, y') + \varepsilon/3 > f(x_{n_k}, y_{n_k})$$
Putting these together, we have

\[ f(x_{n_k}, y) > f(x, y) - \epsilon/3 > f(x, y') + \epsilon/3 > f(x_{n_k}, y_{n_k}) \]

This is a contradiction, since \( f(x_{n_k}, y_{n_k}) = g(x_{n_k}) = \sup_{y} \{ f(x_{n_k}, y) \} \).

4

We will show that \( g'(x_0) = f''(x_0)/2 \) by directly evaluating the limit of difference quotients. We have

\[
\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0) - (x - x_0)f'(x)}{(x - x_0)^2}
\]

Note that both the numerator and the denominator of the above expression converge to 0 as \( x \to x_0 \). Since \( f \) is twice differentiable at \( x_0 \), it must be once differentiable in some neighbourhood of \( x_0 \), otherwise the second derivative would not even make sense. Thus we can apply L’Hospital’s rule; the derivative of the numerator is \( f'(x) - f'(x_0) \), while the derivative of the denominator is \( 2(x - x_0) \). In other words, we have

\[
\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{2(x - x_0)} = \frac{f''(0)}{2}
\]

In particular, the limit exists, i.e. \( g \) is differentiable at \( x_0 \).

5

\( f \) is integrable, with integral 0. Note that any closed interval \([x, y]\) with \( x < y \) contains a point \( z \) with \( f(z) = 0 \). Since \( f \) is non-negative, this implies that for any partition \( P \), we have \( L(f, P) = 0 \).

Let \( \epsilon > 0 \). We will find a partition \( P \) with the upper Riemann sum \( U(f, P) < 2\epsilon \), which will prove the result. Consider the function \( g : [\epsilon, 1] \to \mathbb{R} \), which is equal to \( f \) restricted to the interval \( [\epsilon, 1] \). \( g \) has only finitely many points of discontinuity, namely, the finitely many points of the form \( 1/n > \epsilon \) for
\( n \in \mathbb{N} \). Hence by Rudin Theorem 6.10, \( g \) is integrable. Since all lower Riemann Sums of \( g \) are zero, we must have
\[
\int_{\epsilon}^{1} g(x) = 0
\]

In particular, there exists a partition \( P \) of \([\epsilon, 1]\) with \( U(g, P) < \epsilon \).

Now consider the partition of \([0, 1]\) defined by \( P' = P \cup \{0\} \). Then all but the first term of \( U(f, P') \) is contained in \( U(g, P) \). More precisely, we have
\[
U(f, P') = (\sup_{x \in [0, \epsilon]} f(x))(\epsilon - 0) + U(g, P) < \epsilon + \epsilon = 2\epsilon.
\]

Which proves the result.

6

Since \( f : [0, 1] \to \mathbb{R} \) is integrable, it is bounded, i.e. \( |f(x)| < M \) for all \( x \in [0, 1] \). We may assume \( M > 1 \). Let \( \epsilon > 0 \). Let \( \delta < \epsilon/(2M) \).

Note that \( 0 < 1 - \delta < 1 \), and so by Rudin Theorem 3.20 \( \lim_{n \to \infty} (1 - \delta)^n = 0 \).

Let \( N \) be sufficiently large that \( n > N \implies (1 - \delta)^n < \delta \). Then for any \( 0 \leq x \leq 1 - \delta \) and any \( n > N \), we have \( x^n < \delta \). Hence for \( n > N \), we have
\[
\left| \int_0^{1-\delta} f(x)x^n dx \right| \leq \int_0^{1-\delta} |f(x)|x^n dx < \int_0^{1-\delta} M\delta dx < M\delta < \frac{\epsilon}{2}
\]

On the other hand \( x^n \leq 1 \) for \( x \in [0, 1] \), and so
\[
\left| \int_{1-\delta}^{1} f(x)x^n dx \right| < \int_{1-\delta}^{1} |f(x)|x^n dx < \int_{1-\delta}^{1} Mdx = M\delta < \frac{\epsilon}{2}
\]

Putting these together, we have
\[
\left| \int_0^{1} f(x)x^n dx \right| \leq \int_0^{1-\delta} f(x)x^n dx + \int_{1-\delta}^{1} f(x)x^n dx < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
Thus $\lim_n \int_0^1 f(x)x^n dx = 0$. 