Problem set 3    due Nov. 18

(1) In the MLA notes, §3, Exercise 7.
(2) In the MLA notes, §4, Exercise 5.
(3) Let \( V \) be a 3-dimensional vector space, \( \langle v, w \rangle \) an inner product on \( V \) and \( \Omega \in \bigwedge^3(V^*) \), \( \Omega \neq 0 \).

(a) Given \( \mu \in \bigwedge^2(V^*) \) show that there exists a unique vector, \( v_\mu \in V \), such that for all \( \ell \in V^* \):
\[
\mu \wedge \ell = \ell(v_\mu)\Omega. \quad (**)
\]

_Hint:_ It’s clear that \( \mu \wedge \ell = c_\ell \Omega \) for some constant, \( c_\ell \), depending on \( \ell \). Show that this constant depends linearly on \( \ell \). Then show that there exists a unique vector \( v_\mu \in V \) with the property:
\[
c_\ell = \ell(v_\mu)
\]
for all \( \ell \in V^* \).

(b) For \( v \in V \), let \( \ell_v \in V^* \) be the linear functional
\[
w \in V \to \langle v, w \rangle.
\]
Show how to define a cross product on \( V \) by requiring that
\[
v_1 \times v_2 = v_\mu \Leftrightarrow \mu = \ell_{v_1} \wedge \ell_{v_2}.
\]
Show that this cross product is linear in \( v_1 \) and \( v_2 \) and satisfies \( v_1 \times v_2 = -v_2 \times v_1 \).

(c) Let \( V = \mathbb{R}^3 \). Show that if \( \langle v, w \rangle \) is the Euclidean inner product on \( \mathbb{R}^3 \), \( e_1, e_2, \) and \( e_3 \), the standard basis vectors of \( \mathbb{R}^3 \), and \( \Omega = e_1 \wedge e_2 \wedge e_3 \) the standard volume form, then this cross product is the _standard_ cross product.

(4) Let \( U \) be an open subset of \( \mathbb{R}^3 \) and let
\[
\mu_1 = dx_2 \wedge dx_3 \\
\mu_2 = dx_3 \wedge dx_1
\]
and
\[
\mu_3 = dx_1 \wedge dx_2.
\]

(a) If \( f : U \to \mathbb{R} \) is a function of class \( C^1 \) show that \( df = G_1 dx_1 + G_2 dx_2 + G_3 dx_3 \) where \( G = (G_1, G_2, G_3) = \text{grad} f \).
(b) If $\omega = F_1 \, dx_1 + F_2 \, dx_2 + F_3 \, dx_3$ is a one-form on $U$ of class $C^1$ show that
\[ d\omega = G_1 \mu_1 + G_2 \mu_2 + G_3 \mu_3 \]
where $G = \text{curl} F$.

(c) If $\omega = F_1 \mu_1 + F_2 \mu_2 + F_3 \mu_3$ is a two-form on $U$ of class $C^1$ show that
\[ d\omega = g \, dx_1 \wedge dx_2 \wedge dx_3 \]
where $g = \text{div}(F)$. 