Lecture 16

4 Multi-linear Algebra

4.1 Review of Linear Algebra and Topology

In today’s lecture we review chapters 1 and 2 of Munkres. Our ultimate goal (not today) is to develop vector calculus in $n$ dimensions (for example, the generalizations of grad, div, and curl).

Let $V$ be a vector space, and let $v_i \in V, i = 1, \ldots, k$.

1. The $v_i’s$ are linearly independent if the map from $\mathbb{R}^k$ to $V$ mapping $(c_1, \ldots, c_k)$ to $c_1v_1 + \ldots + c_kv_k$ is injective.
2. The $v_i’s$ span $V$ if this map is surjective (onto).
3. If the $v_i’s$ form a basis, then $\dim V = k$.
4. A subset $W$ of $V$ is a subspace if it is also a vector space.
5. Let $V$ and $W$ be vector spaces. A map $A : V \rightarrow W$ is linear if $A(c_1v_1 + c_2v_2) = c_1A(v_1) + c_2A(v_2)$.
6. The kernel of a linear map $A : V \rightarrow W$ is
   $$\ker A = \{v \in V : Av = 0\}.$$ (4.1)
7. The image of $A$ is
   $$\text{Im } A = \{Av : v \in V\}.$$ (4.2)
8. The following is a basic identity:
   $$\dim \ker A + \dim \text{Im } A = \dim V.$$ (4.3)
9. We can associate linear mappings with matrices. Let $v_1, \ldots, v_n$ be a basis for $V$, and let $w_1, \ldots, w_m$ be a basis for $W$. Let
   $$Av_j = \sum_{i=1}^m a_{ij}w_j.$$ (4.4)
   Then we associate the linear map $A$ with the matrix $[a_{ij}]$. We write this $A \sim [a_{ij}]$.
10. If $v_1, \ldots, v_n$ is a basis for $V$ and $u_j = \sum a_{ij}w_j$ are $n$ arbitrary vectors in $W$, then there exists a unique linear mapping $A : V \rightarrow W$ such that $Av_j = u_j$. 


11. Know all the material in Munkres section 2 on matrices and determinants.

12. The quotient space construction. Let $V$ be a vector space and $W$ a subspace. Take any $v \in V$. We define $v + W \equiv \{v + w : w \in W\}$. Sets of this form are called $W$-cosets. One can check that given $v_1 + W$ and $v_2 + W$,

(a) If $v_1 - v_2 \in W$, then $v_1 + W = v_2 + W$.
(b) If $v_1 - v_2 \notin W$, then $(v_1 + W) \cap (v_2 + W) = \emptyset$.

So every vector $v \in V$ belongs to a unique $W$-coset.

The quotient space $V/W$ is the set of all $W$-cosets.

For example, let $V = \mathbb{R}^2$, and let $W = \{(a, 0) : a \in \mathbb{R}\}$. The $W$-cosets are then vertical lines.

The set $V/W$ is a vector space. It satisfies vector addition: $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$. It also satisfies scalar multiplication: $\lambda(v + W) = \lambda v + W$. You should check that the standard axioms for vector spaces are satisfied.

There is a natural projection from $V$ to $V/W$:

$$\pi : V \rightarrow V/W, \; v \rightarrow v + W. \quad (4.5)$$

The map $\pi$ is a linear map, it is surjective, and $\ker \pi = W$. Also, $\text{Im} \; \pi = V/W$, so

$$\dim V/W = \dim \text{Im} \; \pi = \dim V - \dim \ker \pi = \dim V - \dim W. \quad (4.6)$$

### 4.2 Dual Space

13. The dual space construction: Let $V$ be an $n$-dimensional vector space. Define $V^*$ to be the set of all linear functions $\ell : V \rightarrow \mathbb{R}$. Note that if $\ell_1, \ell_2 \in V^*$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, then $\lambda_1 \ell_1 + \lambda_2 \ell_2 \in V^*$, so $V^*$ is a vector space.

What does $V^*$ look like? Let $e_1, \ldots, e_n$ be a basis of $V$. By item (9), there exists a unique linear map $e_i^* \in V^*$ such that

$$\begin{cases} e_i^*(e_i) = 1, \\
 e_i^*(e_j) = 0, \text{ if } j \neq i. \end{cases}$$

Claim. The set of vectors $e_1^*, \ldots, e_n^*$ is a basis of $V^*$.

Proof. Suppose $\ell = \sum c_i e_i^* = 0$. Then $0 = \ell(e_j) = \sum c_i e_i^*(e_j) = c_j$, so $c_1 = \ldots = c_n = 0$. This proves that the vectors $e_i^*$ are linearly independent. Now, if $\ell \in V^*$ and $\ell(e_i) = c_j$ one can check that $\ell = \sum c_i e_i^*$. This proves that the vectors $e_i^*$ span $V^*$. \qed
The vectors $e_1^*, \ldots, e_n^*$ are said to be a \textit{basis of} $V^*$ \textit{dual to} $e_1, \ldots, e_n$.

Note that $\dim V^* = \dim V$.

Suppose that we have a pair of vector spaces $V, W$ and a linear map $A : V \to W$. We get another map
\[ A^* : W^* \to V^*, \tag{4.7} \]
defined by $A^* \ell = \ell \circ A$, where $\ell \in W^*$ is a linear map $\ell : W \to \mathbb{R}$. So $A^* \ell$ is a linear map $A^* \ell : V \to \mathbb{R}$. You can check that $A^* : W^* \to V^*$ is linear.

We look at the matrix description of $A^*$. Define the following bases:
\begin{align*}
  e_1, \ldots, e_n & \text{ a basis of } V \\  f_1, \ldots, f_n & \text{ a basis of } W \\  e_1^*, \ldots, e_n^* & \text{ a basis of } V^* \\  f_1^*, \ldots, f_n^* & \text{ a basis of } W^*. \tag{4.8-4.11} \\
\end{align*}

Then
\[
A^* f_j^* (e_i) = f_j^* (A e_i) = f_j^* (\sum_k a_{ki} f_k) = a_{ji}, \tag{4.12}
\]

So,
\[
A^* f_j = \sum_k a_{jk} e_k^*, \tag{4.13}
\]

which shows that $A^* \sim [a_{ji}] = [a_{ij}]^t$, the transpose of $A$. 

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