3.2 Riemann Integral of Several Variables

Last time we defined the Riemann integral for one variable, and today we generalize to many variables.

**Definition 3.3.** A rectangle is a subset $Q$ of $\mathbb{R}^n$ of the form

$$Q = [a_1, b_1] \times \cdots \times [a_n, b_n],$$

where $a_i, b_i \in \mathbb{R}$.

Note that $x = (x_1, \ldots, x_n) \in Q \iff a_i \leq x_i \leq b_i$ for all $i$. The volume of the rectangle is

$$v(Q) = (b_1 - a_1) \cdots (b_n - a_n),$$

and the width of the rectangle is

$$\text{width}(Q) = \sup_i (b_i - a_i).$$

Recall (stated informally) that given $[a, b] \in \mathbb{R}$, a finite subset $P$ of $[a, b]$ is a partition of $[a, b]$ if $a, b \in P$ and you can write $P = \{t_i : i = 1, \ldots, N\}$, where $t_1 = a < t_2 < \ldots < t_N = b$. An interval $I$ belongs to $P$ if and only if $I$ is one of the intervals $[t_i, t_{i+1}]$.

**Definition 3.4.** A partition $P$ of $Q$ is an $n$-tuple $(P_1, \ldots, P_n)$, where each $P_i$ is a partition of $[a_i, b_i]$.

**Definition 3.5.** A rectangle $R = I_1 \times \cdots \times I_n$ belongs to $P$ if for each $i$, the interval $I_i$ belongs to $P_i$.

Let $f : Q \to \mathbb{R}$ be a bounded function, let $P$ be a partition of $Q$, and let $R$ be a rectangle belonging to $P$.

We define

$$m_R f = \inf_R f = \text{g.l.b.} \{f(x) : x \in \mathbb{R}\}$$

$$M_R f = \sup_R f = \text{l.u.b.} \{f(x) : x \in \mathbb{R}\},$$

from which we define the lower and upper Riemann sums,

$$L(f, P) = \sum_R m_R(f) v(R)$$

$$U(f, P) = \sum_R M_R(f) v(R).$$
It is evident that
\[ L(f, P) \leq U(f, P). \] (3.15)

Now, we will take a sequence of partitions that get finer and finer, and we will define the integral to be the limit.

Let \( P = (P_1, \ldots, P_n) \) and \( P' = (P'_1, \ldots, P'_n) \) be partitions of \( Q \). We say that \( P' \) refines \( P \) if \( P'_i \supset P_i \) for each \( i \).

\textbf{Claim.} If \( P' \) refines \( P \), then
\[ L(f, P') \geq L(f, P). \] (3.16)

\textbf{Proof.} We let \( P_j = P'_j \) for \( j \neq i \), and we let \( P_i = P_i \cup \{a\} \), where \( a \in [a_i, b_i] \). We can create any refinement by multiple applications of this basic refinement. If \( R \) is a rectangle belonging to \( P \), then either

1. \( R \) belongs to \( P' \), or
2. \( R = R' \cup R'' \), where \( R', R'' \) belong to \( P' \).

In the first case, the contribution of \( R \) to \( L(f, P') \) equals the contribution of \( R \) to \( L(f, P) \), so the claim holds.

In the second case,
\[ m_Rv(R) = m_R(v(R') + v(R'')) \] (3.17)
and
\[ m_r = \inf_R f \leq \inf_{R'} f, \inf_{R''} f. \] (3.18)

So,
\[ m_R \leq m_{R'}, m_{R''} \] (3.19)

Taken altogether, this shows that
\[ m_Rv(R) \leq m_{R'}v(R') + m_{R''}v(R'') \] (3.20)

Thus, \( R' \) and \( R'' \) belong to \( P' \). \( \Box \)

\textbf{Claim.} If \( P' \) refines \( P \), then
\[ U(f, P') \leq U(f, P). \] (3.21)

The proof is very similar to the previous proof. Combining the above two claims, we obtain the corollary

\textbf{Corollary 2.} If \( P \) and \( P' \) are partitions, then
\[ U(f, P') \geq L(f, P) \] (3.22)
Proof. Define $P'' = (P_1 \cup P'_1, \ldots, P_n \cup P'_n)$. So, $P''$ refines $P$ and $P'$. We have shown that

$$
U(f, P'') \leq U(f, P)
$$

$$
L(f, P') \leq L(f, P'')
$$

$$
L(f, P'') \leq U(f, P').
$$

(3.23)

Together, these show that

$$
U(f, P) \geq L(f, P').
$$

(3.24)

With this result in mind, we define the lower and upper Riemann integrals:

$$
\int_Q f = \sup_P L(f, P)
$$

$$
\int_Q f = \inf_P U(f, P).
$$

(3.25)

Clearly, we have

$$
\int_Q f \leq \overline{\int}_Q f,
$$

(3.26)

Finally, we define Riemann integrable.

**Definition 3.6.** A function $f$ is *Riemann integrable over $Q$* if the lower and upper Riemann integrals coincide (are equal).

### 3.3 Conditions for Integrability

Our next problem is to determine under what conditions a function is (Riemann) integrable.

Let’s look at a trivial case:

**Claim.** Let $F : Q \rightarrow \mathbb{R}$ be the constant function $f(x) = c$. Then $f$ is R. integrable over $Q$, and

$$
\int_Q c = cv(Q).
$$

(3.27)

**Proof.** Let $P$ be a partition, and let $R$ be a rectangle belonging to $P$. We see that $m_R(f) = M_R(f) = c$, so

$$
U(f, P) = \sum_R M_R(f)v(R) = c \sum_R v(R)
$$

$$
= cv(Q).
$$

(3.28)
Similarly,
\[
L(f, P) = cv(Q). \tag{3.29}
\]

**Corollary 3.** Let \( Q \) be a rectangle, and let \( \{Q_i : i = 1, \ldots, N\} \) be a collection of rectangles covering \( Q \). Then
\[
v(Q) \leq \sum v(Q_i). \tag{3.30}
\]

**Theorem 3.7.** If \( f : Q \to \mathbb{R} \) is continuous, then \( f \) is \( R \). integrable over \( Q \).

**Proof.** We begin with a definition

**Definition 3.8.** Given a partition \( P \) of \( Q \), we define
\[
\text{mesh width}(P) = \sup_R \text{width}(R). \tag{3.31}
\]

Remember that
\[
Q \text{ compact } \implies f : Q \to \mathbb{R} \text{ is uniformly continuous.} \tag{3.32}
\]

That is, given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( x, y \in Q \) and \( |x - y| < \delta \), then \( |f(x) - f(y)| < \epsilon \).

Choose a partition \( P \) of \( Q \) with mesh width less than \( \delta \). Then, for every rectangle \( R \) belonging to \( P \) and for every \( x, y \in R \), we have \( |x - y| < \delta \). By uniform continuity we have, \( M_R(f) - m_R(f) \leq \epsilon \), which is used to show that
\[
U(f, P) - L(f, P) = \sum_R (M_R(f) - m_R(f))v(R)
\]
\[
\leq \epsilon \sum v(R)
\]
\[
\leq \epsilon v(Q). \tag{3.33}
\]

We can take \( \epsilon \to 0 \), so
\[
\sup_P L(f, P) = \inf_P U(f, P), \tag{3.34}
\]
which shows that \( f \) is integrable. \( \square \)

We have shown that continuity is sufficient for integrability. However, continuity is clearly not necessary. What is the general condition for integrability? To state the answer, we need the notion of *measure zero*.

**Definition 3.9.** Suppose \( A \subseteq \mathbb{R}^n \). The set \( A \) is of *measure zero* if for every \( \epsilon > 0 \), there exists a countable covering of \( A \) by rectangles \( Q_1, Q_2, Q_3, \ldots \) such that \( \sum_i v(Q_i) < \epsilon \).
Theorem 3.10. Let $f : Q \to \mathbb{R}$ be a bounded function, and let $A \subseteq Q$ be the set of points where $f$ is not continuous. Then $f$ is R. integrable if and only if $A$ is of measure zero.

Before we prove this, we make some observations about sets of measure zero:

1. Let $A, B \subseteq \mathbb{R}^n$ and suppose $B \subset A$. If $A$ is of measure zero, then $B$ is also of measure zero.

2. Let $A_i \subseteq \mathbb{R}^n$ for $i = 1, 2, 3, \ldots$, and suppose the $A_i$'s are of measure zero. Then $\bigcup A_i$ is also of measure zero.

3. Rectangles are not of measure zero.

We prove the second observation:

For any $\epsilon > 0$, choose coverings $Q_{i,1}, Q_{i,2}, \ldots$ of $A_i$ such that each covering has total volume less than $\epsilon/2^i$. Then $\{Q_{i,j}\}$ is a countable covering of $\bigcup A_i$ of total volume

$$\sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon. \quad (3.35)$$