Solution for 18.112 ps 1

1(Prob1 on P11).
Solution:
\[ |a| < 1, \quad |b| < 1 \implies (1 - a\bar{a})(1 - b\bar{b}) < 1 \]
\[ \implies 1 - a\bar{a} - b\bar{b} + a\bar{a}b\bar{b} < 1 \]
\[ \implies 1 + a\bar{a}b\bar{b} - a \bar{a} - a\bar{b} > a\bar{a} + b\bar{b} - a\bar{b} - \bar{a}b \]
\[ \implies (1 - \bar{a}b)(1 - \bar{a}b) > (a - b)(\bar{a} - \bar{b}) \]
\[ \implies \left| \frac{a - b}{1 - \bar{a}b} \right| < 1. \]

2(Prob4 on P11).
Solution:
• If there is a solution, then
\[ 2|c| = |z - a| + |z + a| \]
\[ \geq |(z - a) - (z + a)| \]
\[ = 2|a|, \]
i.e.
\[ |c| \geq |a|. \]
On the other hand, if
\[ |c| \geq |a|, \]
take
\[ z_0 = \frac{|c|}{|a|} a, \]
then it is easy to check that \( z_0 \) is a solution. Thus the largest value of \(|z|\) is \(|c|\), with corresponding \( z = z_0 \).
Use fundamental inequality and formula (8) on page 8, we can get

\[ 4|c|^2 = (|z + a| + |z - a|)^2 \]
\[ \leq 2(|z + a|^2 + |z - a|^2) \]
\[ = 4(|z|^2 + |a|^2) \]
\[ \implies |z| \geq \sqrt{|c|^2 - |a|^2}, \]

which can be obtained with

\[ z = i \frac{\sqrt{|c|^2 - |a|^2}}{|a|} a. \]

N.B. Geometrically,

\[ |z - a| + |z + a| = 2|c| \]
represents a ellipse, with long axis \(|c|\) and focus \(a\). So the short axis is

\[ \sqrt{|c|^2 - |a|^2}, \]
and thus

\[ \sqrt{|c|^2 - |a|^2} \leq |z| \leq |c|. \]

3(Prob 1 on P17).
Solution: Suppose

\[ az + \bar{b}z + c = 0 \]
is a line, then it has at least two different solutions, say, \(z_0, z_1\). Thus,

\[ az_0 + \bar{b}z_0 + c = 0, \quad az_1 + \bar{b}z_1 + c = 0 \]
\[ \implies a(z_0 - z_1) = b(\bar{z}_1 - \bar{z}_0) \]
\[ \implies |a| = |b|. \]

Thus \(a \neq 0\) and there is a \(\theta\) such that \(b = ae^{i\theta}\).

So

\[ az + \bar{b}z + c = 0 \]
\[ \iff az + ae^{i\theta} \bar{z} + c = 0 \]
\[ \iff z + e^{i\theta} \bar{z} + c/a = 0 \]
\[ \iff e^{-i\frac{\theta}{2}}z + e^{-i\frac{\theta}{2}}\bar{z} + e^{-i\frac{\theta}{2}}c/a = 0. \]
This equation has solution if and only if
\[ e^{-i\frac{\theta}{2}}c/a \in \mathbb{R}, \]
in which case the equation does represent a line, given by
\[ 2\text{Re}(e^{-i\frac{\theta}{2}z}) = -e^{-i\frac{\theta}{2}}c/a. \]

Note that
\[ e^{-i\frac{\theta}{2}}c/a \in \mathbb{R} \]
\[ \iff e^{-i\frac{\theta}{2}c/a} = e^{-i\frac{\theta}{2}}c/a \]
\[ \iff c/(ae^{i\theta}) = c/a \]
\[ \iff c/b = c/a. \]
So the condition in form of \(a, b, c\) is
\[ |a| = |b| \text{ and } c/b = c/a. \]

4(Prob 5 on P17). (We need to suppose \(|a| \neq 1\).)
Solution: Let \(P, Q\) be the points on the plane corresponding to \(a\) and \(1/\bar{a}\). By
\[ \frac{1}{\bar{a}} = \frac{a}{|a|^2} \]
we know that \(O, P, Q\) are on the same line. Suppose the circle intersect the unit circle at points \(R, S\). (They Do intersect at two points!) Then
\[ |\overline{OR}|^2 = 1 = |a||1/\bar{a}| = |\overline{OP}||\overline{OQ}|. \]
By elementary planar geometry, \(\overline{OR}\) tangent to the circle through \(P, Q\), i.e. the radii to the point of intersection are perpendicular. So the two circles intersect at right angle.