Solution for 18.112 ps 4

1(Prob 3 on P130).

**Method 1.** We only need to prove that these functions has no limit as $z$ tends to infinity. We can prove this by constructing two sequence $\{z_n\}$ and $\{w_n\}$ of complex numbers such that

$$\lim_{n \to \infty} z_n = \lim_{n \to \infty} w_n = \infty$$

but

$$\lim_{n \to \infty} f(z_n) \neq \lim_{n \to \infty} f(w_n).$$

- For $f(z) = e^z$, take $z_n = n$, $w_n = -n$;
- For $f(z) = \sin z$ or $f(z) = \cos z$, take $z_n = 2\pi n$, $w_n = 2\pi n + 1$.

**Method 2.** By definition, we only need to prove that

$$\lim_{z \to 0} z^m f\left(\frac{1}{z}\right) \neq 0$$

for any (fixed) $m \in \mathbb{N}$. We can prove this by choosing one sequence $z_n \to 0$ such that

$$\lim_{n \to \infty} z_n^m f\left(\frac{1}{z_n}\right) \neq 0.$$

- For $f(z) = e^z$,
take \[ z_n = 1/n. \]

- For \( f(z) = \sin z \) or \( f(z) = \cos z \), take \[ z_n = \frac{1}{ni}. \]

**Method 3.** In Midterm we proved that if \( f(z) \) is analytic in \( \mathbb{C} \) and has a nonessential singularity at \( \infty \), then \( f \) is a polynomial. Now all the functions

\[ f_1(z) = e^z, \ f_2(z) = \sin z, \ f_3(z) = \cos z \]

are analytic in \( \mathbb{C} \), and none of them is polynomial (by Taylor expansion or by the number of zero points), so they have essential singularities at \( \infty \).

2(Prob 4 on P133).

**Solution:** By the conditions we know that

\[ f(z) = f(0) + zh(z), \]

where \( h(z) \) is analytic in a neighborhood of 0, and

\[ h(0) \neq 0. \]

Thus there is a small neighborhood \( B_\varepsilon(0) \) such that \( h \) is analytic and nonzero in it. By Corollary 2 on Page 142, we can define a single-valued analytic function

\[ \tilde{h}(z) = h(z)^{1/n} \]

on \( B_\varepsilon(0) \). Let

\[ g(z) = z\tilde{h}(z^n), \]

we get

\[ f(z^n) = f(0) + z^n h(z^n) \]

\[ = f(0) + g(z)^n \]

in \( B_\varepsilon(0) \).
Remark: We can drop the condition

\[ f'(0) \neq 0. \]

Since 0 is always a zero point of \( f(z) - f(0) \), we can either write

\[ f(z) = f(0) + z^m h(z), \]

where \( h(z) \) is analytic in a neighborhood of 0, and \( h(0) \neq 0 \); or have

\[ f(z) \equiv f(0). \]

In the first case we can proceed as before with

\[ g(z) = z^m \tilde{h}(z^n), \]

and the second case is trivial.

3(Prob 4 on P148).
Solution: Apply Corollary 2 in page 142 to analytic function

\[ f(z) = z, \]

we see that single-valued analytic branch of \( \log z \) can be defined in any simply connected region which does not contain the origin. Then we can define single-valued analytic branch of \( z^\alpha \) and \( z^z \) by

\[ z^\alpha = e^{\alpha \log z} \]

and

\[ z^z = e^{z \log z}. \]