Lecture 2

Cauchy integral formula again. $U$ an open bounded set in $\mathbb{C}$, $\partial U$ smooth, $f \in C^\infty(\overline{U})$, $z \in U$

$$f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(\eta)}{\eta - z} d\eta + \frac{1}{2\pi i} \int_U \frac{\partial f(\eta)}{\partial \eta} \frac{1}{\eta - z} d\eta \wedge d\bar{\eta}$$

the second term becomes 0 when $f$ is holomorphic, i.e. the area integral vanishes, and we get

$$f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(\eta)}{\eta - z} d\eta$$

Now take $D : |z - a| < \epsilon$, $f \in \mathcal{O}(D) \cap C^\infty(\overline{D})$, then

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + \epsilon e^{i\theta}) d\theta$$

More applications:

**Theorem (Maximum Modulus Principle).** $U$ any open connected set in $\mathbb{C}$, $f \in \mathcal{O}(U)$ then if $|f|$ has a local maximum value at some point $a \in U$ then $f$ has to be constant.

First, a little lemma.
Lemma. If \( f \in \mathcal{O}(U) \) and \( \text{Re} f \equiv 0 \), then \( f \) is constant.

Proof. Trivial consequence of the definition of holomorphic. \( \square \)

Proof of Maximum Modulus Principle. Assume \( f(a) \) is positive (we can do this by a trivial normalization operation). Let \( u(z) = \text{Re} f \). Now from above

\[
f(a) = \frac{1}{2\pi} \int_{0}^{2\pi} f(a + \epsilon e^{i\theta}) d\theta
\]

The LHS is real valued and trivially

\[
f(a) = \frac{1}{2\pi} \int_{0}^{2\pi} f(a) d\theta
\]

we subtract the above 2 and we get

\[
0 = \int_{0}^{2\pi} f(a) - u(a + \epsilon e^{i\theta}) d\theta.
\]

When \( \epsilon \) is sufficiently small, since \( a \) is a local maximum, the integral is greater than 0, \( f(a) = u(a + \epsilon e^{i\theta}) \) so \( \text{Re} f \) is constant in a neighborhood of \( a \) and we can normalize and assume \( \text{Re} f = 0 \) near \( a \), so by analytic continuation \( f \) is constant on \( U \).

Inhomogeneous CR Equation

Consider \( U \) an open bounded subset of \( \mathbb{C} \), \( \partial U \) a smooth boundary, \( g \in C^\infty(U) \). The Inhomogeneous CR equation is the following PDE: find \( f \in C^\infty(U) \) such that

\[
\frac{\partial f}{\partial \bar{z}} = g
\]

The question is, does there exists a solution for arbitrary \( g \)?

First, consider another, simpler version of CR with \( g \in C^\infty_0(\mathbb{C}) \). Does there exists \( f \in C^\infty(\mathbb{C}) \) such that \( \partial f/\partial \bar{z} = g \)?

Lemma. We claim the function \( f \) defined by the integral

\[
f(z) = \frac{1}{2\pi i} \int \frac{g(\eta)}{\eta - z} d\eta \wedge d\bar{\eta}
\]

is in \( C^\infty(\mathbb{C}) \) and satisfies \( \partial f/\partial \bar{z} = g \).

Proof. Perform the change of variables \( w = z - \eta, dw = -d\eta, d\bar{w} = -d\bar{\eta} \) and \( \eta = z - w \) then the integral above becomes

\[
-\int \frac{g(z-w)}{w} dw \wedge d\bar{w} = f(z)
\]

Now it is clear that \( f \in C^\infty(\mathbb{C}) \), because if we take \( \partial / \partial z \), we can just keep differentiating under the integral. And now

\[
\frac{\partial f}{\partial z} = -\frac{1}{2\pi i} \int \frac{(\frac{\partial g}{\partial \bar{\eta}})(z-w)}{w} dw \wedge d\bar{w} = \frac{1}{2\pi i} \int \frac{(\frac{\partial g}{\partial \eta})(\eta)}{\eta - z} d\eta \wedge d\bar{\eta}
\]

Let \( A = \text{supp} g \), so \( A \) is compact, then there exists \( U \) open and bounded such that \( \partial U \) is smooth and \( A \subset U \). For \( g \in C^\infty(U) \) write down using the Cauchy integral formula

\[
g(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{g(\eta)}{\eta - z} d\eta + \frac{1}{2\pi i} \int_{U} \frac{\partial g}{\partial \eta}(\eta) d\eta \wedge d\bar{\eta} \eta - z
\]

On \( \partial U \), \( g \) is identically 0, so the first integral is 0. For the second integral we replace \( A \) by the entire complex plane, so

\[
g(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g}{\partial \eta}(\eta) d\eta \wedge d\bar{\eta} \eta - z
\]

which is the expression for \( \partial f/\partial \bar{z} \) \( \square \)
Now, we want to get rid of our compactly supported criterion. Let $U$ be bounded, $\partial U$ smooth and $g \in C^\infty(\overline{U})$, $\frac{\partial}{\partial z} = g$.

Make the following definition

$$f(z) := \frac{1}{2\pi i} \int_{\partial U} \frac{g(\eta)}{\eta - z} d\eta \wedge d\overline{\eta}$$

Take $a \in U$, $D$ an open disk about $a$, $\overline{D} \subset U$. Check that $f \in C^\infty$ on $D$ and that $\partial f / \partial \overline{z} = g$ on $D$. Since $a$ is arbitrary, if we can prove this we are done. Take $\rho \in C^\infty_0(U)$ so that $\rho \equiv 1$ on a neighborhood of $\overline{D}$, then

$$f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{\rho(\eta)g(\eta)}{\eta - z} d\eta \wedge d\overline{\eta} + \frac{1}{2\pi i} \int_{\partial D} (1 - \rho) \frac{g(\eta)}{\eta - z} d\eta \wedge d\overline{\eta}$$

The first term, $I$, is in $C^\infty_0(\mathbb{C})$, so $I$ is $C^\infty$ on $\mathbb{C}$ and $\partial I / \partial \overline{z} = \rho g$ on $\mathbb{C}$ and so is equal to $g|_D$. We claim that $II|_D$ is in $O(D)$. The integrand is $0$ on an open set containing $D$, so $\partial II / \partial \overline{z} = 0$ on $D$.

We conclude that $\partial f(z) / \partial \overline{z} = g(z)$ on $D$. (The same result could have just been obtained by taking a partition of unity)

**Transition to Several Complex Variables**

We are now dealing with $C^n$, coordinatized by $z = (z_1, \ldots, z_n)$, and $z_k = x_k + iy_k$ and $dz_k = dx_k + idy_k$.

Given $U$ open in $\mathbb{C}^n$, $f \in C^\infty(U)$ we define

$$\frac{\partial f}{\partial z_k} = \frac{1}{2} \left( \frac{\partial f}{\partial x_k} - i \frac{\partial f}{\partial y_k} \right)$$

So the de Rham differential is defined by

$$df = \sum \left( \frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial y_i} dy_i \right) = \sum \frac{\partial f}{\partial z_k} dz_k + \sum \frac{\partial f}{\partial \overline{z}_k} d\overline{z}_k := df + \overline{df}$$

so $df = \partial f + \overline{df}$.

Let $\Omega^1(U)$ be the space of $C^\infty$ de Rham 1-forms, and $u \in \Omega^1(U)$ then

$$u = u' + u'' = \sum a_i dz_i + \sum b_i d\overline{z}_i \quad a_i, b_i \in C^\infty(U)$$

we introduce the following notation

$$\Omega^{1,0} = \left\{ \sum a_k dz_k, a_k \in C^\infty(U) \right\}$$

$$\Omega^{0,1} = \left\{ \sum b_k d\overline{z}_k, b_k \in C^\infty(U) \right\}$$

and therefore there is a decomposition $\Omega^1(U) = \Omega^{1,0}(U) \oplus \Omega^{0,1}(U)$. We can rephrase a couple of the lines above in the following way: $df = \partial f + \overline{df}$, $\partial f \in \Omega^{1,0}$, $\overline{df} \in \Omega^{0,1}$.

**Definition.** $f \in \mathcal{O}(U)$ if $\overline{df} = 0$, i.e. if $\partial f / \partial \overline{z}_k = 0$, $\forall k$.

**Lemma.** For $f, g \in C^\infty(U)$, $\overline{df}g = f \overline{dg} + g\overline{df}$, thus $fg \in \mathcal{O}(U)$.

Obviously, $z_1, \ldots, z_n \in \mathcal{O}(U)$.

If $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{N}$, then $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $z^\alpha \in \mathcal{O}(\mathbb{C})$. Then

$$p(z) = \sum_{|\alpha| \leq N} a_\alpha z^\alpha \in \mathcal{O}(\mathbb{C}^n)$$

Even more generally, suppose we have the formal power series

$$f(z) = \sum_\alpha a_\alpha z^\alpha$$

and $|a_\alpha| \leq CR_1^{-\alpha_1} \cdots R_n^{-\alpha_n}$. Then let $D_k := |z_k| < R_k$ and $D = D_1 \times \cdots \times D_n$ then $f(z)$ converges on $D$ and uniformly on compact sets in $D$, and by differentiation we see that $f \in \mathcal{O}(D)$.

**Definition.** Let $D_k := |z - a_i| < R_n$, then open set $D_1 \times \cdots \times D_n$ is called a **polydisk**.