1. Find the leading term for each of the integrals below for $\lambda >> 1$.

(a) $\int_{-1}^{1} e^{-\lambda x^3} (1 + x^4) \, dx$
(b) $\int_{1}^{\infty} \sqrt{x - 1} e^{-\lambda \cosh x} \, dx$
(c) $\int_{0}^{2} e^{\lambda x (1-x)} \, dx$

2. Find the leading term for each of the integrals below $\lambda >> 1$.

(a) $\int_{-1}^{1} e^{-\lambda x^3} \, dx$
(b) $\int_{1}^{\infty} e^{-\lambda x^2} \, dx$
(c) $\int_{-2}^{1} (\sin x) e^{-\lambda x^2} \, dx$
(d) $\int_{-\pi}^{\pi} e^{-\lambda \sin x} \, dx$
(e) $\int_{0}^{\infty} e^{-\lambda x} e^{-x^2} \, dx$
(f) $\int_{0}^{\lambda} e^{ax} \, dx$
(g) $\int_{0}^{\infty} e^{-\lambda (x+x^5)} \, dx$

3. Find the entire asymptotic series for each of the integrals in problem 2.

Solutions:
In the following, we assume the given integrals to be in the form

$$I(\lambda) = \int_{a}^{b} h(x) e^{-\lambda v(x)} \, dx$$

1. (a) $v(x) = x^3$, which has a minimum at the lower end point $-1$. Since $v(x)$ is monotonically increasing in $[-1, 1]$, we can use the formula

$$I(\lambda) \approx \frac{e^{-\lambda v(a)} h(a)}{\lambda v'(a)}$$

(1)

to obtain the leading term as

$$I(\lambda) \approx \frac{2e^\lambda}{3\lambda}$$
(b) The integral can be written as
\[
I(\lambda) = \int_0^\infty t^{1/2} e^{-\lambda \cosh(t+1)} dt
\]
\[
= \int_0^\infty t^{1/2} e^{-\lambda [\cosh 1 + t \sinh 1 + ...]} dt
\]
\[
\approx e^{-\lambda \cosh 1} \int_0^\infty t^{1/2} e^{-\lambda t \sinh 1} dt
\]
\[
= e^{-\lambda \cosh 1} \int_0^\infty \left( \frac{1}{\lambda \sinh 1} \right)^{3/2} s^{1/2} e^{-s} ds
\]
\[
= e^{-\lambda \cosh 1} (\lambda \sinh 1)^{3/2} \Gamma(3/2)
\]
\[
= e^{-\lambda \cosh 1} \sqrt{\frac{\pi}{2}}
\]

(c) \(v(x) = -x(1-x)\) takes its minimum at \(x = 1/2\), which is an interior point. As \(v''(1/2) \neq 0\), we can use the formula
\[
I(\lambda) \approx \sqrt{\frac{2\pi}{\lambda |v''(x_0)|}} e^{-\lambda v(x_0)} h(x_0)
\]
to obtain the leading term as
\[
I(\lambda) \approx \sqrt{\frac{\pi}{2\lambda}} e^{\lambda/4}
\]

2. (a) \(v(x) = x^3\), which takes its minimum at \(x = -1\). So, by using (1), we find
\[
I(\lambda) \approx \frac{e^\lambda}{3\lambda}
\]
(b) \(v(x) = x^2\), takes its minimum at \(x = 1\). Therefore, using (1), we find
\[
I(\lambda) \approx \frac{e^{-\lambda}}{2\lambda}
\]
(c) \[
\int_{-2}^{1} (\sin x) e^{-\lambda x^2} dx = \int_{-2}^{-1} (\sin x) e^{-\lambda x^2} dx + \int_{-1}^{1} (\sin x) e^{-\lambda x^2} dx
\]
where the second integral is zero, because its integrand is odd. Therefore, we only consider the first integral. \(v(x) = x^2\), which takes its minimum at \(x = -1\) and is monotonically decreasing throughout \([-2, -1]\). Therefore, by using the formula
\[
I(\lambda) \approx -\frac{e^{-\lambda v(b)} h(b)}{\lambda v'(b)}
\]
to obtain the leading term as
\[
I(\lambda) \approx -\frac{e^{-\lambda}}{2\lambda} \sin 1
\]
(d) \( v(x) = \sin x \) takes its minimum at \( x = -\frac{\pi}{2} \), an interior point. Therefore, the formula (2) gives the leading term

\[ \sqrt{\frac{2\pi}{\lambda}} e^\lambda \]

(e) \( h(x) = e^{-x^2} \) and \( v(x) = x \), which takes its minimum at \( x = 0 \) and is monotonic throughout the domain of integration. Therefore, the relevant formula is (3), which gives

\[ I(\lambda) \approx \frac{1}{\lambda} \]

(f) Since the main contribution comes from \( x = \lambda \) part, we can replace the integral by

\[
I(\lambda) = \int_1^\lambda e^{x^3} \, dx = \int_1^\lambda \frac{1}{3x^2} 3x^2 e^{x^3} \, dx = \frac{1}{3} e^{x^3} \bigg|_1^\lambda + \int_1^\lambda \frac{2}{3} x e^{x^3} \, dx
\]

which implies the leading term

\[ \frac{1}{3\lambda^2} e^{\lambda^3/3} \]

(g) \( v(x) = x + x^5 \), which takes on its minimum at \( x = 0 \), therefore by using the formula (3), we obtain

\[ \frac{1}{\lambda} \]

3. (a) We first let \( s = x^3 + 1 \), then the integral becomes

\[
I(\lambda) = e^\lambda \int_0^2 e^{-\lambda s} \frac{1}{3} (1 - s)^{2/3} \, ds
\]

where now the contribution comes from \( s = 0 \). So we can change the upper limit to \( \infty \). We further let \( \rho = \lambda s \), to obtain

\[
I(\lambda) = e^\lambda \int_0^\infty e^{-\rho} \left( \frac{\rho}{\lambda} + 1 \right)^{2/3} \, d\rho
\]

The idea behind all those transformations is to have the leading term \( \frac{e^\lambda}{3\lambda} \) outside the integral, as above. Now we expand, and get

\[
I(\lambda) = \frac{e^\lambda}{3\lambda} \int_0^\infty e^{-\rho} \sum_{k=0}^\infty (-1)^k \frac{\Gamma(2/3)}{k! \Gamma(2/3 - k)} \left( \frac{\rho}{\lambda} \right)^k \, d\rho
\]

We illegitimately change the order of integration and summation, to obtain the asymptotic series

\[
I(\lambda) = \frac{e^\lambda}{3\lambda} \sum_{k=0}^\infty \frac{1}{k! \Gamma(2/3)} \frac{\Gamma(2/3 + k)}{\Gamma(2/3)} \frac{1}{\lambda^k} \int_0^\infty e^{-\rho} \rho^k \, d\rho
\]

\[
= \frac{e^\lambda}{3\lambda} \sum_{k=0}^\infty \frac{\Gamma(2/3 + k)}{\Gamma(2/3)} \frac{1}{\lambda^k}
\]
(b) We first let \( s = x^2 - 1 \), then the integral becomes

\[
I(\lambda) = e^{-\lambda} \int_{0}^{2} e^{-\lambda s} \frac{1}{2}(s + 1)^{-1/2} ds
\]

where now the contribution comes from \( s = 0 \). So we can change the upper limit to \( \infty \). We further let \( \rho = \lambda s \), to obtain

\[
I(\lambda) = \frac{e^{-\lambda}}{2\lambda} \int_{0}^{\infty} e^{-\rho}(1 + \rho)^{-1/2} d\rho
\]

The idea behind all those transformations is to have the leading term \( \frac{\lambda}{2\lambda} \) outside the integral, as above. Now we expand, and get

\[
I(\lambda) = \frac{e^{-\lambda}}{2\lambda} \int_{0}^{\infty} e^{-\rho} \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k \frac{\Gamma(1/2 + k)}{\Gamma(1/2)} (\frac{\rho}{\lambda})^k d\rho
\]

We illegitimately change the order of integration and summation, to obtain the asymptotic series

\[
I(\lambda) = \frac{e^{-\lambda}}{2\lambda} \sum_{k=0}^{\infty} \frac{\Gamma(1/2 + k)}{\Gamma(1/2)} (-1)^k \frac{1}{\lambda^k}
\]

(c) We consider only

\[
I(\lambda) = \int_{-1}^{0} (\sin x) e^{-\lambda x^2} dx
\]

first let \( s = x + 1 \), to obtain

\[
I(\lambda) = \int_{-1}^{0} \sin(s - 1) e^{-\lambda[s^2 - 2s + 1]} ds \approx e^{-\lambda} \int_{-\infty}^{0} \sin(s - 1) e^{-\lambda s^2} ds
\]

then we further let \( \rho = -2\lambda s \), to obtain

\[
I(\lambda) \approx e^{-\lambda} \int_{0}^{\infty} \sin(1 + (\frac{\rho}{2\lambda})) e^{-\rho} e^{-\rho^2/4\lambda} d\rho
\]

Then, expanding

\[
\sin(1 + (\frac{\rho}{2\lambda})) = \sin 1 + \frac{1}{1!} \cos 1(\frac{\rho}{2\lambda}) - \frac{1}{2!} \sin 1(\frac{\rho}{2\lambda})^2 + ...
\]

and

\[
e^{-\rho^2/4\lambda} = \sum_{k=0}^{\infty} (\frac{-\rho^2}{4\lambda})^k
\]

and plugging those in, one may obtain the entire asymptotic series of the given integral.

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We first let \( s = x + \frac{\pi}{2} \), as the main contribution comes from \( x = -\frac{\pi}{2} \). This gives

\[
I(\lambda) \approx \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{-\lambda \cos s} ds \approx \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\lambda \cos s} ds = 2 \int_{0}^{\frac{\pi}{2}} e^{-\lambda \cos s} ds
\]

As a second step, we let \( \rho = -\lambda (\cos s - 1) \), to obtain

\[
I(\lambda) \approx 2e^\lambda \int_{0}^{1} e^{-\rho} \left( \frac{\rho}{\lambda} \right)^{-1/2} (2 - \frac{\rho}{\lambda})^{-1/2} d\rho \approx \frac{e^\lambda}{(2\lambda)^{1/2}} \int_{0}^{1} e^{-\rho} \rho^{-1/2} (1 - \frac{\rho}{2\lambda})^{-1/2} d\rho
\]

Changing illegitimately, the order of integration and summation, we obtain

\[
I(\lambda) \approx \sqrt{2} e^\lambda \int_{0}^{\infty} \frac{1}{\lambda^{1/2}} \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{1}{2\lambda} \right)^k \Gamma(1/2 + k) \Gamma(1/2) \int_{0}^{1} d\rho e^{-\rho} \rho^{k-1/2}
\]

The asymptotic series is obtained by replacing the upper limit of the integral by \( \infty \), and it is

\[
\sqrt{2\pi} e^\lambda \int_{0}^{\infty} \frac{1}{\lambda^{1/2}} \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{1}{2\lambda} \right)^k \Gamma(1/2 + k) \Gamma(1/2) 1_{\lambda} (-1)^k \Gamma(2/3 + k) \Gamma(2/3)
\]

Letting \( \rho = \lambda x \), we get

\[
I(\lambda) = \frac{1}{\lambda} \int_{0}^{\infty} e^{-\rho} e^{-\left( \frac{\rho}{\lambda} \right)^2} d\rho
\]

\[
= \frac{1}{\lambda} \int_{0}^{\infty} e^{-\rho} \sum_{k=0}^{\infty} \frac{1}{k!} (-\frac{\rho}{\lambda})^{2k} d\rho
\]

and so the asymptotic series is

\[
\frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^{2k} \int_{0}^{\infty} d\rho e^{-\rho} \rho^{2k} = \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{(2k)!}{k!} \frac{(-1)^{2k}}{1^{2k}}
\]

We first let \( s = x^3 - \lambda^3 \), to obtain

\[
I(\lambda) = e^{\lambda^3} \int_{-\lambda}^{0} e^{s} \frac{1}{3}(s + \lambda^3)^{-2/3} ds
\]

\[
= \frac{e^{\lambda^3}}{3\lambda^2} \int_{-\lambda}^{0} e^{s} \left( 1 + \frac{s}{\lambda^3} \right)^{-2/3} ds
\]

\[
= \frac{e^{\lambda^3}}{3\lambda^2} \int_{-\lambda}^{0} e^{s} \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{s}{\lambda^3} \right)^k (-1)^k \frac{\Gamma(2/3 + k)}{\Gamma(2/3)} ds
\]

Therefore the asymptotic series is

\[
\frac{e^{\lambda^3}}{3\lambda^2} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma(2/3 + k)}{\Gamma(2/3)} \frac{1}{\lambda^{3k}} (-1)^k \int_{-\infty}^{0} s^{k} e^{s} ds
\]

\[
= \frac{e^{\lambda^3}}{3\lambda^2} \sum_{k=0}^{\infty} \frac{\Gamma(2/3 + k)}{\Gamma(2/3)} \frac{1}{\lambda^{3k}}
\]
(g) We let \( s = \lambda x \), to obtain

\[
I(\lambda) = \frac{1}{\lambda} \int_{0}^{\infty} e^{-s} e^{-s^5/\lambda^4} ds \\
= \frac{1}{\lambda} \int_{0}^{\infty} e^{-s} \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k \frac{s^{5k}}{\lambda^{4k}} ds
\]

Therefore the asymptotic series is

\[
\frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{1}{\lambda^{4k} k!} (-1)^k \int_{0}^{\infty} e^{-s} s^{5k} ds = \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{(5k)!}{k!} (-1)^k \frac{1}{\lambda^{4k}}
\]