21) \[ F(z) = \int_{a}^{b} dx \ e^{-i\lambda x} f(x), \quad I(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dj \ e^{i\lambda j} F(j). \]

We evaluate \( I(z) \) for \( a < z < b \) as follows.

\[ I(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dj \ e^{i\lambda j} \int_{a}^{b} dx' \ e^{-i\lambda x'} f(x'). \]

In order to interchange the order of integration, we deform the path in \( x' \) to the lower \( z' \)-plane \( (z' = x' + iy') \) for \( \lambda > 0 \) and to the upper \( z' \)-plane for \( \lambda < 0 \):

\[ I(z) = \frac{1}{2\pi} \left[ \int_{C_2} dj \ e^{i\lambda j} \int_{C_2} dz' \ e^{-i\lambda z'} f(z') 
+ \int_{C_1} dj \ e^{i\lambda j} \int_{C_1} dz' \ e^{-i\lambda z'} f(z') \right] \]

\[ = \frac{1}{2\pi} \left[ \int_{C_2} dz' \ f(z') \int_{-\infty}^{0} dj \ e^{i(\lambda-z')j} + \int_{C_1} dz' \ f(z') \int_{0}^{\infty} dj \ e^{i(\lambda-z')j} \right] \]

The exponent in the first integral is \( i(\lambda-z')j \) with \( j > 0 \) and \( \text{Im}(\lambda-z') < 0 \), hence, \( \text{Re}[i(\lambda-z')j] < 0 \) and the integral converges absolutely.

The exponent in the second integral is \( i(\lambda-z')j \) with \( j > 0 \) and \( \text{Im}(\lambda-z') > 0 \), hence, \( \text{Re}[i(\lambda-z')j] < 0 \) and the integral converges absolutely.
\[ I(z) = \frac{1}{2\pi i} \left[ \int_{C_2} dz' f(z') \frac{1}{i(z-z')} + \int_{C_1} dz' f(z') \frac{1}{i(z'-z)} \right] \]

\[ = \frac{1}{2\pi i} \left[ \int_{C_2} dz' \frac{f(z')}{z-z'} + \int_{C_1} dz' \frac{f(z')}{z'-z} \right] = \frac{1}{2\pi i} \int_{C_1 \cup C_2} dz' \frac{f(z')}{z'-z} \]

\[ \Rightarrow I(z) = \frac{1}{2\pi i} \int_{C_1 \cup C_2} dz' \frac{f(z')}{z-z'} = f(z), \quad a < z < b, \]

by the Cauchy integral theorem, since \( f(z) \) is analytic inside \( C_1 \cup C_2 \) and along \( C_1 \cup C_2 \).

Clearly, if \( z < a \) or \( z > b \),

\[ I(z) = 0. \]
\[ \int_{0}^{\infty} K_0(x-y) \cdot u(y) \, dy = 2\pi, \quad x > 0 \]

Let
\[ g(x) = \begin{cases} 0, & x > 0 \\ \int_{0}^{\infty} K_0(x-y) \cdot u(y) \, dy, & x < 0, \end{cases} \quad h(x) = \begin{cases} 2\pi, & x > 0, \\ 0, & x < 0, \end{cases} \]

while \( u(x) = 0, \quad x < 0 \). The given IE is extended to \(-\infty < x < \infty\) as
\[ \int_{-\infty}^{\infty} K_0(x-y) \cdot u(y) \, dy = g(x) + h(x), \quad -\infty < x < \infty. \]

The FT of this equation gives
\[ \tilde{\tilde{g}}(f) + \tilde{\tilde{h}}(f) = \tilde{\tilde{f}}(f), \quad f : \text{real}, \tag{1} \]

where
\[ \tilde{\tilde{g}}(f) = \int_{-\infty}^{\infty} dt \, e^{-i f t} \cdot K_0(t) = \frac{2\pi}{\sqrt{1 + f^2}}, \]
\[ \tilde{\tilde{h}}(f) = \int_{0}^{\infty} dx \, e^{-i f x} \cdot 2\pi = \frac{2\pi}{i f}, \quad \text{Im} f < 0 : \text{a } '+' \text{ function}, \]
and
\[ \tilde{\tilde{g}}(f) = \int_{-\infty}^{0} dx \, e^{-i f x} g(x) : \text{an unknown } '+' \text{ function}. \]

In order to be able to work with \( f : \text{real} \) we "move" the pole of \( \tilde{\tilde{h}}(f) \) to the upper half plane by \( i \varepsilon \):
\[ \tilde{\tilde{h}}(f) = \lim_{\varepsilon \to 0^+} \frac{2\pi}{i(f-i\varepsilon)}. \]

Then Eq. (1) reads
\[ \frac{2\pi}{\sqrt{1 + f^2}} \cdot \tilde{\tilde{f}}(f) = \tilde{\tilde{g}}(f) + \frac{2\pi}{i(f-i\varepsilon)} \quad (1a) \]
We factorize \( \frac{2\pi}{\sqrt{i\epsilon^2}} \) as
\[
2\pi \frac{1}{\sqrt{\frac{1}{J+i}}} \frac{1}{\sqrt{\frac{1}{J-i}}} + -
\]
where the branch cuts for the branch points \( J=\pm i \) are chosen as shown here:

\[ j\text{-plane} \]

Eq. (1a) is written as
\[
\frac{2\pi}{\sqrt{i\epsilon^2}} \tilde{u}(J) = \sqrt{\frac{1}{J+i}} \tilde{g}(J) + \frac{2\pi}{i(J-i\epsilon)} \sqrt{\frac{1}{J+i}} (1b)
\]

We further decomposed the "mixed" \((\oplus)\) last term to + and - parts:
\[
\frac{2\pi}{i(J-i\epsilon)} \sqrt{\frac{1}{J+i}} = \frac{2\pi}{i(J-i\epsilon)} \left( \sqrt{\frac{1}{J+i}} - \sqrt{\frac{1}{i\epsilon+i}} \right) + \frac{2\pi}{i(J-i\epsilon)} \sqrt{\frac{1}{i\epsilon+i}}
\]

- no pole at \( J=\epsilon i \);
- only a pole at \( J=\epsilon i \);
- function
- function

Equation (1b) is equal to
\[
\frac{2\pi}{\sqrt{J-i}} \tilde{u}(J) - \frac{2\pi}{i(J-i\epsilon)} \sqrt{i\epsilon+i} = \sqrt{\frac{1}{J+i}} \tilde{g}(J) + \frac{2\pi}{i(J-i\epsilon)} \left( \sqrt{\frac{1}{J+i}} - \sqrt{\frac{1}{i\epsilon+i}} \right)
\]

\[ \equiv E(J) : \text{entire function} \]

From the left-hand side of this equation, it follows that
\( E(J) \to 0 \) as \( J \to \infty \) for \( \text{Im} J < 0 \); hence \( E(J) \equiv 0 \) in the lower half plane.

From the right-hand side, it then follows that \( E(J) \equiv 0 \) everywhere.
Thus,

\[ 0 = \frac{2\pi}{\sqrt{j-i}} \tilde{u}(j) - \frac{2\pi}{i(j-\epsilon)} \sqrt{i(\epsilon)} = \sqrt{j+i} \tilde{g}(j) + \frac{2\pi}{i(j-\epsilon)} \left[ \sqrt{j+i} - \sqrt{i(\epsilon)} \right] \]

Solve for \( \tilde{u}(j) \):

\[ \tilde{u}(j) = \sqrt{i(\epsilon)} \frac{\sqrt{j-i}}{i(j-i)} \]

Invert to get \( u(x) \):

\[ u(x) = \int_{-\infty}^{\infty} \frac{d\tilde{u}}{2\pi} \ e^{ix} \tilde{u}(j) = \int_{-\infty}^{\infty} \frac{d\tilde{u}}{2\pi} \ e^{ix} \sqrt{j-i} \frac{i}{i(j-i)} \]

where \( \sqrt{i(\epsilon)} = \sqrt{j+i} \frac{j-i}{j-i} = e^{i\pi/2} \frac{j+i}{j-i} \frac{\epsilon^{1/2}}{\epsilon^{-1/2}} = e^{i\pi/4} \).

Clearly \( u(x) = 0 \) for \( x < 0 \) by closing the path in the lower h.p.

\[ u(x) = e^{i\pi/4} \int_{-\infty}^{\infty} \frac{d\tilde{u}}{2\pi} \ e^{ix} \sqrt{j-i} \]

the path is below the pole at \( j = 0 \).

Let \( j' = j - i \):

\[ u(x) = e^{i\pi/4} \int_{-\infty}^{\infty} \frac{d\tilde{u}}{2\pi} \ e^{-x} \ e^{ix} \sqrt{j'} \]

\[ \frac{\partial u}{\partial x} = \frac{i}{e^{i\pi/4}} \int_{-\infty}^{\infty} \frac{d\tilde{u}}{2\pi} \ e^{ix} \sqrt{j'} \]: the integrand has no pole at \( j' = -i \).

We wrap the contour around the branch cut emanating from \( j' = 0 \).
\[
\frac{\partial u}{\partial x} = i e^{i n \frac{\pi}{2}} e^{-x} \int_{0}^{\infty} \frac{d(y)}{2\pi} e^{-y x} \sqrt{y} e^{i n \frac{\pi}{2} y} = \frac{1}{2} i e^{i n \frac{\pi}{2}} e^{-x} \int_{0}^{\infty} \frac{d(y)}{2\pi} e^{-y x} \sqrt{y} e^{i (n + \frac{\pi}{2}) y}
\]

(set \( y = e^{i \frac{\pi}{2}} \) on right
side of branch cut)

\[
= \frac{1}{2} i e^{-x} \int_{0}^{\infty} \frac{d(y)}{2\pi} e^{-y x} \sqrt{y} = i e^{-x} \int_{0}^{\infty} \frac{dy}{2\pi} e^{-y x} \sqrt{y}
\]

(set \( y = e^{i (\frac{\pi}{2} + 2n)} \) on
left side of branch cut)

\[
= -2 e^{-x} \int_{0}^{\infty} \frac{dy}{2\pi} e^{-y x} \sqrt{y} = \frac{1}{2} e^{-\frac{1}{x}} \int_{0}^{\infty} \frac{dt}{\sqrt{t}} e^{-t \cdot \frac{3}{2} + 1}
\]

\[
= \frac{1}{\sqrt{2\pi}} \frac{e^{-x}}{x^{\frac{3}{2}}}
\]

So,

\[
\frac{\partial u}{\partial x} = -\frac{1}{2\sqrt{\pi}} \frac{e^{-x}}{x^{\frac{3}{2}}}, \quad x > 0
\]

while

\[
\frac{\partial u}{\partial x} = u(x + \infty) = 0.
\]

Hence,

\[
u(x) = \int_{0}^{x} dx' \frac{\partial u}{\partial x'} = \frac{1}{2\sqrt{\pi}} \int_{0}^{x} dx' \frac{e^{-x'}}{x'^{\frac{3}{2}}} = \frac{(-2)}{2\sqrt{\pi}} \int_{x}^{\infty} \frac{d(x' - \frac{1}{2})}{x'} e^{-x'}
\]

\[
= \frac{-1}{\sqrt{\pi}} \left[ -\frac{e^{-x}}{x^{\frac{1}{2}}} + \int_{x}^{\infty} \frac{d(x' - \frac{1}{2})}{x'} e^{-x'} \right] = \frac{-1}{\sqrt{\pi}} \left[ -\frac{e^{-x}}{x^{\frac{1}{2}}} + 2 \int_{x}^{\infty} \frac{d(x' - \frac{1}{2})}{x'} e^{-x'} \right]
\]

\[
= \frac{e^{-x}}{\sqrt{\pi} x} - \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} \frac{dz}{z} e^{-z^2} = \text{erfc}(\sqrt{x})
\]

\[
\approx \text{erfc}(\sqrt{x}) \text{; } \text{"error function," known from probability theory}
\]

\[
\therefore \quad u(x) = \frac{e^{-x}}{\sqrt{\pi x}} - \text{erfc}(\sqrt{x}), \quad x > 0
\]
\[ \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{i\xi x} \chi(\xi) F(\xi) = f(x), \quad x > 0, \]
\[ \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{i\xi x} F(\xi) = g(x), \quad x < 0. \]

Define
\[ H(x) = \begin{cases} 0 & , \quad x > 0, \\ \int_{-\infty}^{0} \frac{d\xi}{2\pi} e^{i\xi x} \chi(\xi) F(\xi) & , \quad x < 0, \end{cases} \]
\[ L(x) = \begin{cases} 0 & , \quad x < 0, \\ \int_{0}^{\infty} \frac{d\xi}{2\pi} e^{i\xi x} F(\xi) & , \quad x > 0. \end{cases} \]

The original dual equations are thus defined for all real \( x \):
\[ \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{i\xi x} \chi(\xi) F(\xi) = f(x) + H(x), \quad -\infty < x < +\infty, \]
\[ \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{i\xi x} F(\xi) = g(x) + L(x), \quad -\infty < x < +\infty. \]

By taking the FT of both sides of these equations we get
\[ \chi(\xi) F(\xi) = \hat{\chi}(\xi) + \hat{H}(\xi) \]
\[ F(\xi) = \hat{g}(\xi) + \hat{L}(\xi) \]
\[ \Rightarrow \chi(\xi) \left[ \hat{g}(\xi) + \hat{L}(\xi) \right] = \hat{f}(\xi) + \hat{H}(\xi), \]
and \( \xi \) is assumed to be on the real axis or on a line parallel to the real axis. Let us definitely assume that \( \xi \) lies on the real axis.
We invert the last relation to get
\[ \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{i\xi x} \mathcal{X}(\xi) \left[ \tilde{g}(\xi) + \tilde{h}(\xi) \right] = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} e^{i\xi x} \left[ \tilde{f}(\xi) + \tilde{h}(\xi) \right], \]
or
\[ \int_{-\infty}^{\infty} dx' K(x-x') g(x') + \int_{-\infty}^{\infty} dx' K(x-x') \cdot L(x') = f(x) + H(x) \]
For \( x > 0 \), \( H(x) = 0 \) and
\[ \int_{-\infty}^{\infty} dx' K(x-x') \cdot L(x') = f(x) - \int_{-\infty}^{0} dx' K(x-x') g(x'), \]
or,
\[ \int_{-\infty}^{\infty} dx' K(x-x') \cdot L(x') = f(x) + g_1(x), \quad g_1(x) = -\int_{-\infty}^{0} dx' K(x-x') g(x'), \]
where \( L(x) \) is unknown, and \( f(x) \) and \( g_1(x) \) are known. Hence, the two original integral equations reduce to a single integral equation which is a Wiener-Hopf equation of the first kind. This last equation can be solved by the method of factorization. For this purpose, we return to the original relation among FTs,
\[ \mathcal{X}(\xi) \left[ \tilde{g}(\xi) + \tilde{L}(\xi) \right] = \tilde{f}(\xi) + \tilde{H}(\xi), \quad \xi: \text{real}. \]
By setting
\[ \mathcal{X}(\xi) = \frac{P_{-}(\xi)}{P_{+}(\xi)}, \]
we get
\[ P_{-}(\xi) \tilde{L}(\xi) = P_{+}(\xi) \tilde{f}(\xi) - P_{-}(\xi) \tilde{g}(\xi) + P_{+}(\xi) \tilde{H}(\xi). \]
Since \( \tilde{f}(j) \) and \( \tilde{g}(j) \) are known, we decompose the RHS as

\[
\begin{align*}
\mathcal{P}_+(j) \tilde{f}(j) - \mathcal{P}_-(j) \tilde{g}(j) & \equiv R_+(j) + R_-(j), \\
- & +
\end{align*}
\]

which in turn entails

\[
\mathcal{P}_+(j) \tilde{L}(j) - \mathcal{P}_-(j) = R_+(j) + A(j) \tilde{H}(j) \equiv E(j), \quad j: \text{real}
\]

entire for

The function \( E(j) \) is determined via taking the limits of

\( \tilde{P}_- \tilde{L} - R_- \) and \( R_+ + \tilde{P}_+ \tilde{H} \) as \( |j| \to \infty \) in the respective half planes, using

the fact that \( \tilde{H}, \tilde{L} \to 0 \) as \( |j| \to \infty \). Finally,

\[
\tilde{L}(j) = \frac{E(j) + R_-(j)}{\mathcal{P}_-(j)}, \quad \tilde{H}(j) = \frac{E(j) - R_+(j)}{\mathcal{P}_+(j)}.
\]

Note: The procedure given here merely shows that the dual integral equations reduce to a single integral equation of the first kind. Hence, the solvability issues that "plague" equations of the first kind are still inherent in the dual integral equations. One case where the dual equations are explicitly solved is studied in connection with the mixed boundary-value problem of Laplace's equation. (Compare with class notes and verify this statement!)