1.1 Statement: Haberman problem 57.06.

Consider an infinite number of cars, each designated by a number $\beta$. Assume that the car labeled by $\beta$ starts from $x = \beta$ ($\beta > 0$) with zero velocity, and also assume it has a constant acceleration $\beta$.

(a) Determine the position and velocity of each car as a function of time.
(b) Sketch the path of a typical car.
(c) Determine the velocity field \( u = u(x, t) \).
(d) Sketch the curves along which \( u = u(x, t) \) is constant.

1.2 Statement: Haberman problem 60.01.

Consider a semi-infinite highway \( 0 \leq x < \infty \) (with no entrances of exits other than at \( x = 0 \)). Show that the number of cars on the highway at time \( t \) is:

\[
N_0 + \int_0^t q(0, \tau) d\tau, \tag{1.1}
\]

where \( N_0 \) is the number of cars in the highway at time \( t = 0 \). You may assume that \( \rho(x, t) \to 0 \) as \( x \to \infty \). Justify the equation both: directly (by physical reasoning), as well as by using the equation \( \rho_t + q_x = 0 \).

1.3 Statement: Haberman problem 60.03.

(a) Without any mathematics, explain why \( \int_{a(t)}^{b(t)} \rho(x, t) \, dx \) is constant if \( a \) and \( b \) (not equal to each other) are moving with the traffic.

(b) Using part (a), re-derive the equation

\[
\frac{d}{dt} \int_a^b \rho(x, t) \, dx = q(a, t) - q(b, t), \tag{1.2}
\]

where \( a < b \) are any two points in the road.

(c) Assuming \( \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u) \), verify mathematically that part (a) is valid.

1.4 Statement: Haberman problem 60.04.

If the traffic flow is increasing as \( x \) increases (\( \frac{\partial q}{\partial x} > 0 \)), explain physically\(^1\) why the density must be decreasing in time (\( \frac{\partial \rho}{\partial t} < 0 \)).

\(^1\)You do not need any equations.
1.5 Statement: Haberman problem 67.04.

Consider the equation

\[(\rho_1)_t + c(\rho_1)_x = 0. \tag{1.3}\]

Suppose that we observe \(\rho_1\) in a coordinate system moving at velocity \(v\). Show that

\[(\rho_1)_t + (c - v)(\rho_1)_x = 0. \tag{1.4}\]

Does the car density \(\rho\) stay constant moving at the car velocity?

1.6 Statement: Linear 1st order PDE # 08.

Solve the problem below, using the method of characteristics: (a) Compute the characteristics, as done in the lectures, starting from each point in the data set. (b) Next solve for the solution \(u\) along each characteristic. (c) Finally, eliminate the characteristic variables \(\zeta\) and \(s\) from the expression\(^2\) for \(u\) obtained in step (b) — using the result in step (a) — to obtain the solution as a function of \(x\) and \(y\).

\[(x - y)u_x + (x + y)u_y = x^2 + y^2, \tag{1.5}\]

with data \(u(x, 0) = (1/2)x^2\) for \(1 \leq x < \exp(2\pi)\).

**Further question:** Where in the \((x, y)\) plane does the problem above define the solution \(u\)? That is: what is the region of the plane characterized by the property that: through each point in this region there is exactly one characteristic connecting it with the curve where the data is given?

2 Special Problems.

2.1 Statement: Linear 1st order PDE # 02.

Consider the following problem

\[xu_x + yu_y = 1 + y^2; \quad \text{with} \quad u(x, 1) = 1 + x \quad \text{for} \quad -\infty < x < \infty. \tag{2.6}\]

**Part 1.** Use the method of characteristics to solve this problem. Write the solution \(u = u(x, y)\) (explicitly!) as a function of \(x\) and \(y\) on \(y > 0\). **Hint.** Write the characteristic equations: \(\frac{dx}{ds} = \ldots, \frac{dy}{ds} = \ldots, \text{and} \frac{du}{ds} = \ldots\). Then solve these equations using the initial data (for \(s = 0\)) \(x = \tau, y = 1, \text{and} \ u = 1 + \tau, \text{for} \ -\infty < \tau < \infty\). Finally, eliminate \(s\) and \(\tau\), to get \(u\) as a function of \(x\) and \(y\).

**Part 2.** Explain why \(u = u(x, y)\) is not determined by the problem above for \(y \leq 0\) (you may use a diagram). **Hint.** Draw, in the \(x-y\) plane, the characteristic curves computed in part 1.

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\(^2\)Here \(\zeta\) is the label for each characteristic, and \(s\) is a parameter along the characteristics.
2.2 Statement: Linear 1st order PDE # 09 (surface evolution).

The evolution of a material surface can (sometimes) be modeled by a pde. In evaporation dynamics, where the material evaporates into the surrounding environment, consider a surface described in terms of its “height” $h = h(x, y, t)$ relative to the $(x, y)$-plane of reference. Under appropriate conditions, a rather complicated pde can be written for $h$. Here we consider a (drastically) simplified version of the problem, where the governing equation is

$$h_t = \frac{A}{r} h_r, \quad \text{for } r = \sqrt{x^2 + y^2} > 0 \text{ and } t > 0, \quad \text{where } A > 0 \text{ is a constant.} \quad (2.7)$$

Axial symmetry is assumed, so that $h = h(r, t)$. Obviously, $h$ should be an even function of $r$. This is both evident from the symmetry, and necessary in the equation to avoid singular behavior at the origin. Assume now

$$h(r, 0) = H(r^2), \quad (2.8)$$

where $H$ is a smooth function describing a localized bump. Specifically: (i) $H(0) > 0$, (ii) $H$ is monotone decreasing, (iii) $H \to 0$ as $r \to \infty$. Note that $h(r, 0)$ is an even function of $r$.

1. Using the theory of characteristics, write an explicit formula for the solution of $(2.7 - 2.8)$.
2. Do a sketch of the characteristics in space time — i.e.: $r > 0$ and $t > 0$.
3. What happens with the characteristic starting at $r = \zeta > 0$ and $t = 0$ when $t = \zeta^2/2A$?
4. Show that the resulting solution is an even function of $r$ for all times.
5. Show that, as $t \to \infty$, the bump shrinks and vanishes. Hint: pick some example function $H$ with the properties above, and plot the solution for various times. This will help you figure out why the bump shrinks and vanishes.

THE END.

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3 From mass conservation, with the details of the physics going into modeling the flux and sink/source terms.