1. (a) (5 points) Solve the recurrence $f(0) = 2$, $f(1) = 4$, and

$$f(n + 2) = 4f(n + 1) - 2f(n), \quad n \geq 0.$$ 

(Give a simple explicit formula for $f(n)$. It is o.k. to use irrational numbers.)

(b) (5 points) Show that $\lfloor (2 + \sqrt{2})^n \rfloor$ is odd for all integers $n \geq 0$.

2. (10 points) Let $f(n)$ be the number of ways to choose a permutation $\pi$ of $1, 2, \ldots, n$ and color each cycle of $\pi$ of even length either red or blue. For instance, $f(1) = 1$, $f(2) = 3$, $f(3) = 9$, and $f(4) = 45$. Set $f(0) = 1$. Find the generating function

$$F(x) = \sum_{n \geq 0} f(n) \frac{x^n}{n!}.$$ 

To get full credit, your answer should not involve any infinite sums, infinite products, or the functions exp (the exponential function) and log.

3. (a) (5 points) Let $f(n)$ be the number of ways to tile a $2 \times n$ rectangle with $1 \times 1$ squares and $2 \times k$ rectangles for any integer $k \geq 1$, where the $2 \times 1$ rectangle must be vertical. (Set $f(0) = 1$.) For instance, $f(2) = 5$, given by

A larger example of such a tiling is given by
Find a simple expression for the generating function

\[ F(x) = \sum_{n \geq 0} f(n)x^n. \]

(b) (5 points; difficult) Let \( g(n) \) be the number of ways to tile a \( 2 \times n \) rectangle with \( a \times b \) rectangles for any integers \( a, b \geq 1 \). (Set \( g(0) = 1 \).) For instance, \( g(2) = 8 \), given by

A larger example of such a tiling is given by

Find a simple expression for the generating function

\[ G(x) = \sum_{n \geq 0} g(n)x^n. \]

4. (10 points) Let \( m, n \geq 3 \). Let \( G \) be the graph obtained by identifying an edge of an \( m \)-cycle with the edge of an \( n \)-cycle. Thus \( G \) has \( m + n - 2 \) vertices and \( m + n - 1 \) edges. Find the number \( \kappa(G) \) of spanning trees of \( G \). (It is easiest to use “naive” reasoning and not to use the Matrix-Tree Theorem.) The figure below shows the case \( m = 5 \) and \( n = 4 \).
5. (10 points) Let $G$ be a regular bipartite graph of degree $d \geq 2$, i.e., every vertex of $G$ has the same degree $d \geq 2$. Show that $G$ has a spanning subgraph (that is, a subgraph using every vertex of $G$) that is a disjoint union of cycles. (No two cycles should have a vertex in common.)

6. (10 points) Compute the chromatic polynomial of the following graph $G$ with eight vertices:

![Graph with eight vertices]

7. (a) (5 points) Does there exist a planarly embedded graph with no isthmus (i.e., no edge $e$ for which the same face lies on both sides of $e$) and with exactly one face with $k$ vertices, $3 \leq k \leq 8$, and with no other faces? Thus $G$ has six faces in all.

(b) (5 points) Same question, but for $3 \leq k \leq 9$.

8. (10 points) Find the least positive integer $n$ with the following property: if the edges of $K_n$ are colored red and blue, then there must exist a monochromatic non-closed path of length three, i.e., a path of length three (not a triangle, so with four vertices and three edges) such that all edges in the path have the same color. (Five points for showing that some value of $n$ has this property. Full credit for finding the least such value and showing that it is indeed least.)
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