Exercises 2

(1) [3] Show that for any arrangement $A$, we have $\chi_{cA}(t) = (t-1)\chi_A(t)$, where $cA$ denotes the cone over $A$. (Use Whitney’s theorem.)

(2) [2–] Let $G$ be a graph on the vertex set $[n]$. Show that the bond lattice $L_G$ is a sub-join-semilattice of the partition lattice $\Pi_n$ but is not in general a sublattice of $\Pi_n$.

(3) [2–] Let $G$ be a forest (graph with no cycles) on the vertex set $[n]$. Show that $L_G \cong B_{E(G)}$, the boolean algebra of all subsets of $E(G)$.

(4) [2] Let $G$ be a graph with $n$ vertices and $A_G$ the corresponding graphical arrangement. Suppose that $G$ has a $k$-element clique, i.e., $k$ vertices such that any two are adjacent. Show that $k!\rho(A)$.

(5) [2+] Let $G$ be a graph on the vertex set $[n] = \{1, 2, \ldots, n\}$, and let $A_G$ be the corresponding graphical arrangement (over any field $K$, but you may assume $K = \mathbb{R}$ if you wish). Let $C_n$ be the coordinate hyperplane arrangement, consisting of the hyperplanes $x_i = 0$, $1 \leq i \leq n$. Express $\chi_{A_G \cup C_n}(t)$ in terms of $\chi_{A_G}(t)$.

(6) [4] Let $G$ be a planar graph, i.e., $G$ can be drawn in the plane without crossing edges. Show that $\chi_{A_G}(4) \neq 0$.

(7) [2+] Let $G$ be a graph with $n$ vertices. Show directly from the the deletion-contraction recurrence (20) that $(-1)^n\chi_G(-1) = \#AO(G)$.

(8) [2+] Let $\chi_G(t) = t^n - c_{n-1}t^{n-1} + \cdots + (-1)^{n-1}c_1t$ be the chromatic polynomial of the graph $G$. Let $i$ be a vertex of $G$. Show that $c_1$ is equal to the number of acyclic orientations of $G$ whose unique source is $i$. (A source is a vertex with no arrows pointing in. In particular, an isolated vertex is a source.)

(9) [5] Let $A$ be an arrangement with characteristic polynomial $\chi_A(t) = t^n - c_{n-1}t^{n-1} + c_{n-2}t^{n-2} - \cdots + (-1)^nc_0$. Show that the sequence $c_0, c_1, \ldots, c_n = 1$ is unimodal, i.e., for some $j$ we have $c_0 \leq c_1 \leq \cdots \leq c_j \geq c_{j+1} \geq \cdots \geq c_n$.

(10) [2+] Let $f(n)$ be the total number of faces of the braid arrangement $B_n$. Find a simple formula for the generating function

$$
\sum_{n \geq 0} f(n) \frac{x^n}{n!} = 1 + x + 3 \frac{x^2}{2!} + 13 \frac{x^3}{3!} + 75 \frac{x^4}{4!} + 541 \frac{x^5}{5!} + 4683 \frac{x^6}{6!} + \cdots.
$$

More generally, let $f_k(n)$ denote the number of $k$-dimensional faces of $B_n$. For instance, $f_1(n) = 1$ (for $n \geq 1$) and $f_n(n) = n!$. Find a simple formula for the generating function

$$
\sum_{n \geq 0} \sum_{k \geq 0} f_k(n) y^k \frac{x^n}{n!} = 1 + xy + (y + 2y^2) \frac{x^2}{2!} + (y + 6y^2 + 6y^3) \frac{x^3}{3!} + \cdots.
$$