Thm (Schar) (~1915)
\[\forall r \exists n=n(r) \text{ s.t. in every coloring of } \mathbb{Z}_{[n]} \exists \text{ a monochromatic Schur triple } x, y, x+y^3\]
e.g. \(r=1, n=3\)
\(r=2, n=9 \text{ or something}\)
In fact, \(n(r) \leq e \cdot r!\)

Proof (Sukna's book) (p.326)
\(X: \mathbb{Z}_{[n]} \rightarrow [r]\). Pick the most popular color \(c\),
Let \(A_1 = \{x_0, \ldots, x_m^3 \in X \mid c\}, m! \geq \frac{n}{r}\)
\(B_1 = \{x_1, \ldots, x_m^3 \} \) must have no \(c\), (i.e. \(x_i + 2z = \text{out}\))
\(A_2 \subseteq B_1\), \(c_2\) most popular color in \(B_1\), \(A_2 = X(c_2) \cap B_1\)
\(\frac{n}{r!} \Rightarrow \text{etc. repeat } r \text{ times. Eventually you eliminate all colors but one.} \checkmark\)

Thm (van der Waerden) (~1928)
\(\forall r \exists n=n(r) \text{ s.t. } \forall r\)-coloring of \(\mathbb{Z}_{[n]} \) 
monochromatic arithmetic progression of length \(l\)
I.e. Let \(H_n^l\) be hypergraph on \(n\) vertices
w/edges being length \(l\) arithmetic sequences, then \(\chi(H_n^l) \rightarrow \infty \)
Pf: Induction on \( l \). \( k = \frac{1}{2} \), \( l = 2 \), \( n(K, 2) = k! \)

For all \( n \geq n(K, 2) \). Now, take \( n \) large enough, break \( [n] \) into 
\[ \geq 2m, \text{ intervals of length } t, \quad \exists r^t, \text{ colorings} \]
of each interval. Let \( m = n(r^t, 2) \). So
\[ \exists \text{ arith. seq. of same coloring intervals of length } n \geq 2m \text{.} \]

Now let \( t = 2m_2 t_2 \), divide \( [n] \) into 
\[ 2m, \text{ intervals of length } t_2, \quad m_2 = n(r^t, 2) \]
so \( \exists \) identically colored progression, etc.

Repeat \( r \) times, \( w / t_r = l \), Okay.

Now look at \( a_1, \ldots, a_{l+1}, \ldots, a_{l+2} \), etc.,
i.e., numbers in first slot of all list of intervals
in first of all but last, (in which it's the end), first
of all, but last 2, where it's the end, etc.

This is \( K+1 \), numbers, so two are colored
the same, say \( a_{l+1} \), \( a_{l+2} \), and \( a_{l+1} \).

Then progression is \( a_{l+1}, \ldots, a_{l+2} \), and \( a_{l+1} \).

\[ l \leq i \leq l+1 \]