Instructions: Solve your favourite problems from the list below. Open problems are marked with \*; hard (but feasible) problems are marked with \*.

1. (Széky, 1997) For a simple graph \( G \) and \( k \in \mathbb{N} \), let \( kG \) denote the graph obtained from \( G \) by replacing every edge by \( k \) parallel edges. Show that \( k^2 \cdot \text{cr}(G) = \text{cr}(kG) \).

2. For \( t \in \mathbb{N} \), we are given an alphabet of size \( t \) and a string \( s \) of \( 2^t \) letters. Find a (nonempty) substring \( s' \) of consecutive letters from \( s \) such that each letter occurs in \( s' \) an even number of times.

3. (Djidjev and Vrót, 2003) Given a simple topological graph \( G(V, E, D) \), where \( D \) stands for the planar embedding of the graph, let \( \ell(D) \) be the maximum number of edges crossed by a vertical line. The cut width \( \text{cw}(G) \) of a simple graph \( G(E, V) \) is defined as the minimum \( \ell(D) \) over all drawings \( D \) in which the vertices have distinct \( x \)-coordinates. (a) Show that \( \text{bw}(G) \leq \text{cw}(G) \). (b) Show that

\[
\text{cr}(G) = \Omega(\text{cw}(G)) - O \left( \sum_{p \in V} \deg(p) \right).
\]

4. (a) For every \( k \in \mathbb{N} \), construct a graph whose crossing number is \( k \).
(b) For every \( k \in \mathbb{N} \), find a graph \( G(V, E) \) and an edge \( pq \in E \) such that \( \text{cr}(G) = k \) but \( G'(V, E \setminus \{pq\}) \) is planar.
(c) Find 3-regular graphs \( G(E, V) \) such that \( \text{cr}(G) = 1 \), but \( G'(V, E \setminus \{pq\}) \) is planar for any edge \( pq \in E \).
(d) Every 3-regular graph \( G(V, E) \) has an edge \( pq \in E \) such that the crossing number of \( G'(V, E \setminus \{pq\}) \) is at least \( \Omega(\text{cr}(G)) - O(1) \). *
(e) (Richter and Thomassen, 1993) Show (d) for simple graphs. *

5. \( K \) is a complete geometric graph with \( n \) vertices, each edge is colored red or blue.
(a) (Bialostocki and Dierker) Show that \( K \) contains a monochromatic spanning tree with pairwise non-crossing edges if \( V \) forms the vertex set of a convex \( n \)-gon.
(b) (Károlyi et al., 1997) Show that \( K \) contains a monochromatic spanning tree with pairwise non-crossing edges. *
(c) Show that \( K \) contains \((n+1)/3\) pairwise disjoint edges of the same color.
(d) Color the edges of a complete graph \( K_n \) with two colors such that there are no \((n+1)/3 + 1\) pairwise disjoint edges of the same color.

6. Let \( \text{lin-cr}(G) \) denote the rectilinear crossing number of \( G \), which is the minimum number of crossings in a drawing of \( G \) with all edges drawn as straight line segments.
(a) Show that if \( \text{cr}(G) = 1 \), then \( \text{lin-cr}(G) = 1 \).
(b) Find a simple graph where \( \text{cr}(G) < \text{lin-cr}(G) \). *