18.319, Fall 2005. Problem Set 6

Open problems are marked with ✰; hard (but feasible) problems are marked with $\star$.

1. (Radon, 1952) Every set of $d+2$ points in $\mathbb{R}^d$ can be partitioned into two subsets such that their convex hulls intersect.

2. (Las Vergnas & Lovász, 1972) You are given a finite range space (X, R) such that for every $k \in \mathbb{N}$, the union of every k ranges contains at least $k+1$ elements. Show that there is a 2-coloring of X such that no range is monochromatic

3. (a) (Spencer, 1985) Given a range space (X, R) with $|X| = |R| = n$, show that its discrepancy is $\mathrm{disc}(R) = O(\sqrt{n})$.

   (b) (Beck, 1981) Given the range space (X, R) where $X = \{1, 2, \ldots, n\}$ and R is the set of all arithmetic progressions over X, prove that $\mathrm{disc}(R) = O(n^{1/4} \,\mathrm{polylog}\, n)$. $\star\star$

4. (a) Every set of n disjoint unit disks in the plane has a BSP of size $O(n)$.

   (b) Every set of n disjoint disks in the plane has a BSP of size $O(n)$.

   (c) (de Berg, 2000) For every $d \in \mathbb{N}$, every set of n disjoint fat objects in $\mathbb{R}^d$ has a BSP of size $O(n)$. (An object is *fat* if the ratio of the radii of the smallest enclosing and largest inscribed balls is bounded by a constant.)

5. (a) (Patterson & Yao, 1990) Every set of n pairwise disjoint axis-parallel lines in $\mathbb{R}^3$ has a BSP of size $O(n^{3/2})$. Find n disjoint axis-parallel lines in $\mathbb{R}^d$ such that any BSP for them has size $\Omega(n^{3/2})$.

   (c) The BSP-complexity of n disjoint axis-aligned rectangles in $\mathbb{R}^3$ is $\Theta(n^{3/2})$.

   (d) (Dumitrescu, Mitchell, and Sharir, 2004) The BSP-complexity of n disjoint axis-aligned 2-flats (i.e., 2-dimensional rectangles) in $\mathbb{R}^4$ is $\Theta(n^{5/3})$. $\star$

   (e) The BSP-complexity of n disjoint axis-parallel lines in $\mathbb{R}^d$ is $\Theta(n^{\frac{d}{d-1}})$.

   (f) What is the BSP-complexity of n disjoint axis-aligned 3-flats in $\mathbb{R}^5$; ✰ in $\mathbb{R}^6$? ✰

6. (a) (Pach & Pinchasi, 2005) Every set of n pairwise disjoint line segments in the plane has a subset of $\ell = \Omega(n^{1/3})$ segments that can be completed to a noncrossing simple path of 2ℓ vertices by adding $\ell - 1$ new line segments between their endpoints.

   (b) There is a set of n pairwise disjoint line segments in the plane such that no subset of $\ell = 2\lfloor\sqrt{2n}\rfloor + 1$ segments can be completed to a noncrossing simple path this way.

7. (Aichholzer et al., 2004) The number of pointed pseudo-triangulations is minimized for point sets in convex position.

8. (Kettner et al., 2003) Every finite point set in general position in the plane has a pointed pseudo-triangulation (a) whose maximum vertex degree is five;

   (b) whose maximum face degree is four.