ONE $e = 0.6$ KEPLER ORBIT

according to SE, IE and RK4, when employing the indicated numbers of equal steps.
If you would like to check the performance of your SE, IR and/or RK4 integrators for the 4-variable Kepler system

\[
\begin{align*}
\frac{dx}{dt} &= u \\
\frac{dy}{dt} &= v \\
\frac{du}{dt} &= -x / (x^2 + y^2)^{3/2} \\
\frac{dv}{dt} &= -y / (x^2 + y^2)^{3/2}
\end{align*}
\]

with
\[
\begin{align*}
x(0) &= -1.6 \\
y(0) &= 0 \\
u(0) &= 0 \\
v(0) &= -0.5
\end{align*}
\]

illustrated in the front, here are 6-decimal versions of my own final values of \(x, y, u, v\) at \(t = 2\pi\) obtained using each of the methods and (deliberately insufficient) numbers of steps \(N\) indicated:

<table>
<thead>
<tr>
<th>(N)</th>
<th>(x_{\text{fin}})</th>
<th>(y_{\text{fin}})</th>
<th>(u_{\text{fin}})</th>
<th>(v_{\text{fin}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>-1.915385</td>
<td>0.615995</td>
<td>-0.352637</td>
<td>-0.333777</td>
</tr>
<tr>
<td>2000</td>
<td>-1.801689</td>
<td>0.323027</td>
<td>-0.211429</td>
<td>-0.422452</td>
</tr>
<tr>
<td>4000</td>
<td>-1.714675</td>
<td>0.164835</td>
<td>-0.117051</td>
<td>-0.464083</td>
</tr>
</tbody>
</table>

**Simple Ruler:**

\[
\begin{align*}
\frac{dx}{dt} &= u \\
\frac{dy}{dt} &= v \\
\frac{du}{dt} &= -x / (x^2 + y^2)^{3/2} \\
\frac{dv}{dt} &= -y / (x^2 + y^2)^{3/2}
\end{align*}
\]

\[
\begin{align*}
x(0) &= -1.6 \\
y(0) &= 0 \\
u(0) &= 0 \\
v(0) &= -0.5
\end{align*}
\]

**Improved Ruler** (= average of the end slopes):

<table>
<thead>
<tr>
<th>(N)</th>
<th>(x_{\text{fin}})</th>
<th>(y_{\text{fin}})</th>
<th>(u_{\text{fin}})</th>
<th>(v_{\text{fin}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>-1.756273</td>
<td>0.914366</td>
<td>-0.368226</td>
<td>-0.276509</td>
</tr>
<tr>
<td>100</td>
<td>-1.663363</td>
<td>0.238959</td>
<td>-0.110471</td>
<td>-0.467563</td>
</tr>
<tr>
<td>200</td>
<td>-1.611030</td>
<td>0.052969</td>
<td>-0.022515</td>
<td>-0.496202</td>
</tr>
</tbody>
</table>

**4th-order Runge-Kutta**:

<table>
<thead>
<tr>
<th>(N)</th>
<th>(x_{\text{fin}})</th>
<th>(y_{\text{fin}})</th>
<th>(u_{\text{fin}})</th>
<th>(v_{\text{fin}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>-0.600508</td>
<td>-0.541975</td>
<td>0.996085</td>
<td>-0.362168</td>
</tr>
<tr>
<td>30</td>
<td>-1.449241</td>
<td>-0.163156</td>
<td>0.140100</td>
<td>-0.531470</td>
</tr>
<tr>
<td>40</td>
<td>-1.562383</td>
<td>-0.050097</td>
<td>0.036481</td>
<td>-0.509762</td>
</tr>
</tbody>
</table>

versus -1.6 0.0 0.0 -0.5

Of course those numbers do not yet match the initial quadruplet as they "ought" to have ... but bear in mind that all these calculations employed double precision (= approx. 14 significant digits) internally. Thus any big discrepancies here must have arisen from the step sizes \(dt\) having been chosen too coarsely, and not from round-off errors.

And yes, the hodograph (= locus in the \(u,v\) velocity space) for any Keplerian orbit should be strictly a CIRCLE!!