23 Water waves

If you look out onto the River Charles, the waves that are immediately apparent are surface waves on the water. However, there are many different types of waves in the rivers and oceans, which have profound effects on our surroundings. The most dramatic example is a Tsunami, which is a wave train generated by earthquakes and volcanoes. Before considering these, however, let's begin by considering the motion of a disturbance on the surface of water.

23.1 Deep water waves

The flow is assumed to be inviscid, and as it is initially irrotational it must remain so. Fluid motion is therefore described by the velocity potential $(u, v) = \nabla\phi$, and satisfies Laplace's equation (incompressibility condition)

$$\nabla^2\phi = 0. \tag{541a}$$

The momentum equation becomes

$$\frac{\partial \nabla\phi}{\partial t} + \frac{1}{2}\nabla(\nabla\phi)^2 = -\frac{1}{\rho}\nabla p - \nabla\chi, \tag{541b}$$

where χ is the gravitational potential such that $\mathbf{g} = -\nabla\chi$. This can be integrated to give the unsteady Bernoulli relation

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} + \chi = C(t). \tag{542}$$

Here, $C(t)$ is a time dependent constant that does not affect the flow, which is related to ϕ only through spatial gradients. The surface is $h(x, t)$ and we have the kinematic condition

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = v \tag{543}$$

on $y = h(x, t)$. This simply states that if you choose an element of fluid on the surface, the rate at which that part of the surface rises or falls is, by definition, the vertical velocity. Finally, we require that the pressure be atmospheric, p_0 at the surface. From the unsteady Bernoulli relation we get

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(u^2 + v^2) + gh = 0 \tag{544}$$

on $h(x, t)$, where we have chosen the constant $C(t)$ appropriately to simplify things.

The equations we have derived so far take account of the effect of gravity on the free surface. We have ignored one important factor, however, which is *surface tension*. It costs energy to create waves, as they have greater surface area than a flat surface. From our earlier work we know that a pressure jump exists across a distorted interface. If p_0 is atmospheric pressure, then the pressure at the fluid surface is

$$p = p_0 - \gamma\frac{\partial^2 h(x, t)}{\partial x^2}. \tag{545}$$

Including surface tension in our pressure condition at the surface, we have that

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(u^2 + v^2) + gh - \frac{\gamma}{\rho}\frac{\partial^2 h}{\partial x^2} = 0 \tag{546}$$

at $y = h(x, t)$.

We now follow the same procedure as in the last lecture and assume all the variables to be small, so that we can linearise the equations. The linearised system of equations consists of Laplace's equation and the boundary conditions at $y = 0$:

$$\nabla^2\phi = 0 \tag{547a}$$
$$\frac{\partial h}{\partial t} = \frac{\partial \phi}{\partial y}(x, 0, t), \tag{547b}$$
$$\frac{\partial \phi}{\partial t} = -gh + \frac{\gamma}{\rho}\frac{\partial^2 h}{\partial x^2}, \tag{547c}$$

These conditions arise because we have Taylor expanded terms such as

$$v(x, h, t) = v(x, 0, t) + hv_y(x, 0, t), \tag{548}$$

and then ignored nonlinear terms. We guess solutions of the form

$$\phi = Ae^{ky}\sin(kx - \omega t)\ , \qquad h = \epsilon e^{ky}\cos(kx - \omega t), \tag{549}$$

knowing that these satisfy Laplace's equation (we have ignored terms of the form e^{-ky}, as the surface is at $y = 0$ and we need all terms to disappear as $y \to -\infty$). Putting these into the surface boundary conditions (8) and (9) gives

$$\omega\epsilon = Ak, \tag{550a}$$
$$\omega A = g\epsilon + \frac{\gamma k^2\epsilon}{\rho}. \tag{550b}$$

Eliminating A we get the dispersion relation

$$\omega^2 = gk + \frac{\gamma k^3}{\rho}. \tag{551}$$

What are the consequences of this relation? On the simplest level we know that the *phase speed*, c, of a disturbance is given by the relation $c = \omega/k$. Thus

$$c^2 = \frac{g}{k} + \frac{\gamma k}{\rho}. \tag{552}$$

The relative importance of surface tension and gravity in determining wave motion is given by the *Bond number* $B_o = \gamma k^2/\rho g$. If $B_o < 1$ then we have *gravity waves*, for which longer wavelengths travel faster. If $B_o > 1$ then we have *capillary waves*, for which shorter wavelengths travel faster. For water, the Bond number becomes unity for wavelengths of about 2 cm, and this accounts for the different ring patterns you can observe when a stone and a raindrop fall into water.

23.2 Properties of the dispersion relation

When a group of waves travels across the surface of water each particular wave crest moves faster than the group as a whole, i.e., if you look closely then wave crests within the disturbance appear to move through it. Why is this? The answer is that different Fourier components of the disturbance move at different speeds. Such a system is said to be *dispersive*.

If we consider a stone thrown into a pond, and we take the Fourier transform of the disturbance it creates, then that disturbance is described by

$$h(x,t) = \int_{-\infty}^{\infty} \hat{h}_k \mathrm{e}^{i[kx-\omega(k)t]} dk. \tag{553}$$

Now the dominant wavelength in the disturbance corresponds to the diameter of the rock d. We shall call the corresponding wavenumber $k_0 = 2\pi/d$. Other wavenumbers will also be excited but we argue that $\hat{h}_k$ will be very small except when k is very near to k_0. Near k_0 we have that

$$\omega(k) \approx \omega(k_0) + \omega'(k_0)(k-k_0). \tag{554}$$

where $\omega' = \partial\omega/\partial k$. Thus

$$h(x,t) = \mathrm{e}^{i[k_0 x-\omega(k_0)t]} \int_{-\infty}^{\infty} \hat{h}_k \mathrm{e}^{i(k-k_0)[x-\omega'(k_0)t]} dk. \tag{555}$$

The first term of this expression is a travelling wave moving with $p$hase velocity

$$c_p = \frac{\omega(k_0)}{k_0} \tag{556a}$$

The second term is a function only of $[x-\omega'(k_0)t]$. It corresponds to an envelope moving with the $g$roup velocity

$$c_g = \omega'(k_0) \tag{556b}$$

that encloses the travelling wave describes. Thus, the wave packet as a whole moves with c_g. It is a simple step to recognise that if $\omega(k) \neq ck$ then $c_g \neq c_p$. For gravity waves in deep water, we have

$$c_g = \frac{d}{dk}(gk)^{\frac{1}{2}} = \frac{1}{2}c_p. \tag{557a}$$

Alternatively for capillary waves

$$c_g = \frac{d}{dk}(\gamma k^3/\rho)^{\frac{1}{2}} = \frac{3}{2}c_p. \tag{557b}$$

For comparison, the dispersion relation for sound waves, $\omega = ck$, tells us that

$$h(x,t) = \int_{-\infty}^{+\infty} b_k \mathrm{e}^{ik(x\pm ct)} dk. \tag{558}$$

So we start with an arbitrary disturbance, and this perturbation just moves without changing shape (although in three dimensions there would be a decay in amplitude due to power conservation).

This is not so for water waves, which have a different dispersion relation, and we can highlight the difference between these two cases by considering the wakes behind an airplane and a boat.

23.3 The wake of an airplane

The equation governing the propagation of a 2D disturbance in air is the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi = c^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right), \tag{559}$$

where ϕ is some scalar quantity representing the disturbance (e.g., the velocity potential, the density or the pressure). For an airplane moving through the air we anticipate a solution that is constant in the frame of reference of the plane. Thus

$$\phi(x, y, t) = \tilde{\phi}(x - Ut, y), \tag{560}$$

and we have

$$U^2 \frac{\partial^2 \tilde{\phi}}{\partial x^2} = c^2 \left(\frac{\partial^2 \tilde{\phi}}{\partial x^2} + \frac{\partial^2 \tilde{\phi}}{\partial y^2} \right). \tag{561}$$

Defining the *Mach number* $M = U^2/c^2$, the above equation becomes

$$(1 - M^2) \frac{\partial^2 \tilde{\phi}}{\partial x^2} + \frac{\partial^2 \tilde{\phi}}{\partial y^2} = 0. \tag{562}$$

If $M < 1$ we can make a simple change of variables $X = x/\sqrt{1 - M^2}$ and regain Laplace's equation. Thus everything can be solved using our conformal mapping techniques. However, if $M > 1$ then the original equation now looks like a wave equation, with y replacing t, yielding solutions of the form

$$\tilde{\phi}(x, y) = \Phi(x - y\sqrt{M^2 - 1}) \tag{563}$$

Thus disturbances are confined to a wake whose half angle is given by

$$\tan \theta = \frac{1}{\sqrt{M^2 - 1}}. \tag{564}$$

Only a narrow region behind the plane knows it exists, and the air ahead doesn't know what's coming!

23.4 Flow created by a 1D 'boat'

We now consider a boat moving at constant speed across the surface of water. The motion of the boat generates a disturbance at point (x', t'). The total disturbance generated as the boat progresses is the sum of the individual contributions

$$h(x, t | x', t') = \int dk \, \hat{h}_k e^{i[k(x - x') - \omega(k)(t - t')]} e^{-\Gamma_k (t - t')} \, \Theta(t - t'), \tag{565}$$

where Γ_k describes the attenuation of the disturbance in time. Let us assume the boat's trajectory is given by $x' = Ut'$, corresponding the boat moving from left to right. In this case, the sum over the history of the boat positions is given by

$$\begin{aligned} h(x,t) &\propto \int dx' \int dt'\; h(x,t|x',t')\,\delta(x'-Ut') \\ &= \int dk\,\hat{h}_k e^{i[kx-\omega(k)t]-\Gamma_k t} \int dx' \int dt'\; \hat{h}_k e^{i[-kx'+\omega(k)t']} e^{\Gamma_k t'}\;\Theta(t-t')\,\delta(x'-Ut') \\ &= \int dk\,\hat{h}_k e^{i[kx-\omega(k)t]-\Gamma_k t} \int_{-\infty}^{Ut} dx'\;\hat{h}_k e^{\{-i[(k-\omega(k)/U]+(\Gamma/U)\}x'} \end{aligned} \tag{566}$$

To simplify things a bit, let's focus on spatial perturbations at time $t = 0$

$$\begin{aligned} h(x,0) &\propto \int_{-\infty}^{\infty} dk\;\hat{h}_k e^{ikx} \int_{-\infty}^{0} dx'\; e\{-i[(k-\omega(k)/U]+(\Gamma/U)\}x' \\ &= U\int_{-\infty}^{\infty} dk\;\frac{e^{ikx}}{-i[kU-\omega(k)]+\Gamma_k}. \end{aligned} \tag{567}$$

Now this integral is dominated by Fourier components for which $kU - \omega(k)$ is close to zero. Thus the biggest contribution comes from the component whose phase velocity matches that of the boat. Let's write

$$k = k_0 + \delta k\;, \qquad \omega(k) = \omega(k_0) + \omega'(k_0)\delta k.$$

To further simplify matters we will assume that $\hat{h}_k$ and Γ_k are well approximated by constants $\hat{h}_{k_0}$ and Γ_{k_0} over the range for which the denominator is small. Then, to a good approximation our integral may be expressed as an integral over δk with infinite limits

$$\begin{aligned} h(x,0) &\propto U\hat{h}_{k_0} e^{ik_0x} \int_{-\infty}^{\infty} d(\delta k)\;\frac{e^{i(\delta k)x}}{-iU_0\delta k+\Gamma_{k_0}} \\ &= i\frac{U}{U_0} h_{k_0} e^{ik_0x} \oint_C d(\delta k)\;\frac{e^{i(\delta k)x}}{\delta k + i\Gamma_{k0}/U_0}, \end{aligned} \tag{568}$$

where $U_0 = U - \omega'(k_0)$ is the difference between the boat velocity and the group velocity of the Fourier component, and the contour C includes the real k-axis with other contributions vanishing. If we are considering gravity waves, then U_0 is a positive quantity. The integral has to be evaluated around a contour C in the complex plane. For $x > 0$ there is no pole inside the semicircle and the integral is zero. For $x < 0$, in the lower half of the complex plane there is a pole at $\delta k = i\Gamma_{k_0}/U_0$, and it follows that

$$h(x,0) \propto 2\pi\frac{U}{U_0}\,\hat{h}_{k_0}\,e^{ik_0x}e^{\Gamma k_0 x/U_0}\,\Theta(-x). \tag{569}$$

Thus we see that the boat is trailed by a wave travelling in the same direction, whose wavelength is such that the boat and wave stay in step (i.e., the phase velocity of the wave matches the boat velocity). In front of the boat the amplitude of the wave is zero.

Note that if we had considered the motion of an insect across the water then we would be considering capillary waves. Then the group velocity is faster than the phase velocity. Thus U_0 would be a negative quantity and our complex integration would have revealed a wave that *precedes* the insect, with no disturbance behind it. As mentioned at the start, this analysis is applicable to the steady flow past an obstacle. In this case, if U is the steady stream velocity we can now understand why we see a steady pattern of capillary waves upstream from the object and a steady pattern of gravity waves downstream from the object.

Finally, on open water, the waves created by a boat can move in two dimensions. To describe this, our 1D treatment needs to be extended to account for the V-shaped wake behind a boat, also known as the *Kelvin wedge*. In 2D, the disturbance generated by the boat is

$$h(\boldsymbol{x}, t|\boldsymbol{x}', t') = \int d\boldsymbol{k}\ \hat{h}(\boldsymbol{k}) e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')} e^{i\omega(\boldsymbol{k})(t-t')} \mathrm{e}^{\Gamma_k(t-t')}. \tag{570}$$

As before, the only waves that contribute significantly are those whose phase velocity in the direction of motion of the boat matches the speed of the boat. If the waves are gravity waves, then the relevant $\boldsymbol{k}$-vectors are those with inclination α and magnitude k_α satisfying

$$k_\alpha = \frac{g}{U^2 \sin^2 \alpha}. \tag{571}$$

Turning back to our dispersion relation for water waves, it can readily be shown that the minimum phase velocity is

$$c = \left(\frac{4g\gamma}{\rho}\right)^{1/2}, \tag{572}$$

and this occurs for the wavenumber $k = \sqrt{\rho g/\gamma}$.