First Problem Set


Suggested Problems (textbook):
- Ch. 2: 2.2.9 2.2.12 2.2.13 2.3.3 2.4.9 2.6.1 2.8.3 2.8.5
- Ch. 3: 3.3.1 3.4.5 3.4.7 3.4.8 3.4.9 3.4.10

Problems to hand in for grading (textbook):
- Ch. 2: 2.2.8 2.2.10 2.3.2 2.5.4 2.5.5 2.5.6
- Ch. 3: 3.2.6 3.2.7 3.3.2 3.4.6

Problem to hand in for grading (not in textbook):

PDE_Blow_Up

In the lectures we considered the PDE problem initial value problem:

\[ u_t + uu_x = 0; \quad u(x,0) = F(x). \]

Notation:
1) \( u_t \) and \( u_x \) are the partial derivatives, with respect to \( t \) and \( x \) (resp.).
2) \( t \) is time and \( x \) is space.
3) \( * \) is the multiplication operator.
4) \( ^\circ \) denotes taking a power \( [u^2 \text{ is the square of } u] \).
5) \( u = u(x,t) \) is a function of \( x \) and \( t \).

We showed that the solution to this problem ceased to exist at a finite time (the derivatives of \( u \) become infinite and, beyond that, \( u \) becomes multiple valued) whenever \( dF/dx \) was negative anywhere.

This was shown "graphically". It can be shown analytically as follows:

--- A. Consider the characteristic curves \( dx/dt = u(x,t) \), as introduced in the lecture.

--- B. Along each characteristic curve, one has \( du/dt = 0 \), as shown in class. Now, let \( v = u_x \). Then \( v \) satisfies the equation [obtained by taking the partial derivative with respect to \( x \) of the equation for \( u \)]:

\[ v_t + uu_x + v^2 = 0. \]

Thus, along characteristics: \( dv/dt + v^2 = 0 \). Thus, if \( v \) is negative anywhere, \( v \) develops an infinity in finite time. But the initial conditions for \( v \), along the characteristic such that \( x(0) = x_0 \), is \( v(0) = df/dx(x_0) \).

Hence the conclusion follows: the solution \( u = u(x,t) \) to the problem ceases to exist at a finite time (with the derivative \( u_x \) of \( u \) becoming infinite somewhere) whenever \( df/dx \) is negative anywhere.

Consider now the problem:

\[ u_t + uu_x = -u; \quad u(x,0) = F(x). \]

Show that the solution to this second problem ceases to exist at a finite time, provided that \( df/dx < C < 0 \), where \( C \) is a finite (non-zero) constant. Again, what happens is that the derivatives become infinite. Calculate \( C \).

Hint: Use an approach analogous to the one used above: get an ODE for the derivative \( v = u_x \) along the characteristics, and study the conditions under which the solutions of the ODE blow-up in a finite time.
A graphical approach for how the solution to \( u_t + u^2 u_x = -u \) behaves in time will also work, but the approach using the ODE for \( v \) along characteristics turns out to be simpler.