18.385j/2.036j Problem.

- Balancing a broom.

**Statement:**

Consider the problem of balancing a broom upright, by placing it on a surface that moves up and down in some prescribed manner. Specifically:

Assume a rough flat horizontal surface, which oscillates up and down following some prescribed law (that is, at any time the surface can be described by the equation \( y = Y(t) \), where \( y \) is the vertical coordinate, and \( Y \) is some oscillatory function). On this surface we place a broom, in upright position, with the sweeping side pointing up.\(^2\) **Question:** Can we prescribe \( Y \) in such a way that the broom remains upright — i.e.: the position is stable?

In order to answer the question, consider the following idealized situation:

A) Replace the broom by a mass \( m \), placed at the upper end of a (massless) rigid rod of length \( L \). Let the displacement of the rod from the vertical position be given by the angle \( \theta \), with \( \theta = 0 \) corresponding to the rod standing vertical, and the mass on the upper end.

B) The bottom of the rod is attached to a hinge that allows it to rotate in a plane. Thus the motion of the rod is restricted to occur on a plane.

C) Assume that friction can be neglected.

D) The hinge to which the rod is attached oscillates up and down, with position \( x = 0 \) and \( y = Y(t) \) — \( x \) is the horizontal coordinate on the plane where the rod moves. The mass is then at \( x = L \sin(\theta) \) and \( y = Y + L \cos(\theta) \) — we measure angles clockwise from the top.

**Now, do the following:**

(1) 

Use Newton’s laws to derive the equation of motion for the mass \( m \). You should obtain a second order ODE for the angle \( \theta \), with coefficients depending on the parameters \( g \) (the acceleration of gravity) and the length of the rod \( L \) — in addition to the forcing function \( Y = Y(t) \).

**Hint:** Only two forces act on the mass \( m \), namely: gravity and a force \( F = F(t) \) along the rod. The force \( F \) has just the right strength to keep the (rigid) rod at constant length \( L \) — this is enough to determine \( F \), though you do not need to calculate it.

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\(^2\)Because the surface is rough, the contact point of the broom with the surface will not move relative to the surface.
You should notice that adding a constant velocity to the hinge motion (that is: \( Y \rightarrow Y + vt \), where \( v \) is a constant) does not change the equation of motion. Why should this be so? What physical principle is involved?

Write down the (linearized) equations for small perturbations of the equilibrium position \((\theta = 0)\) that we wish stabilized. Stability occurs if and only if \( Y = Y(t) \) can be selected so that the solutions of this linear equation do not grow in time — strictly speaking we should also consider the possible effects of nonlinearity, but we will ignore this issue here.

You should notice that it is possible to stabilize \( \theta = 0 \) by taking \( Y = -a \, t^2 \), where \( a > 0 \) is a constant acceleration. How large does \( a \) have to be for this to happen? Give a justification of this result based on physical reasoning, without involving any equations (this is something you should have been able to predict before you wrote a single equation).

Of course, the “solution” found in (4) is not very satisfactory, since \( Y \) grows without bound in it. Consider now oscillatory forcing functions of the form:

\[
Y = \ell \cos(\omega t),
\]

where \( \ell > 0 \) and \( \omega > 0 \) are constants (with dimensions of length and time\(^{-1} \), respectively).

The objective is to find conditions on \((\ell, \omega)\) that guarantee stability.

The next steps will lead you through this process, but first: Nondimensionalize the (linearized) stability equation. In doing so it is convenient to use the time scale provided by the forcing to nondimensionalize time — i.e.: let the nondimensional time be \( \tau = \omega t \).

This step should lead you to an equation describing the evolution of the angle \( \theta \) (valid for small angles), involving two nondimensional parameters. One of them, \( \epsilon = \ell/L \), measures the amplitude
of the oscillations in terms of the length of the rod. The other measures the time scale of the forcing (as given by $1/\omega$) in terms of the time scale of the gravitational instability — a function of $g$ and $L$. Call this second parameter $\mu$ — note that in the equation only $\mu^2$ appears, not $\mu$ itself.

Find the stability range for $\mu$ as a function of $\epsilon$, for the values $0 < \epsilon \leq 0.6$ — it is enough to pick a few values of $\epsilon$, say $\epsilon = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$, and then to compute the stability range for each of them.

Note/hint: This step will require not just analysis, but some numerical computation. So as not to be forced to explore all possible values of $\mu$ when looking for the stability ranges (numerically an impossible task), you should notice that the analysis for $\epsilon = 0$ can be done exactly — and should provide you with a good hint as to where to look.

Write the period $p = \frac{2\pi}{\omega}$ of the forcing, in terms of the nondimensional parameter $\mu$, and the parameters $g$ and $L$. The results of part (6) should provide you with the period ranges (for a given oscillation amplitude) where stability occurs. Use this information to provide a rough explanation of why it is relatively easy to balance a broom on the palm of your hand (using the strategy outlined in this problem — try it), and why you will not be able to balance a pencil.

For $0 \leq \epsilon \ll 1$ and $0 \leq \mu \ll 1$ you should be able to obtain analytical approximations for the stable ranges. Do so, and compare your results with those of part (6).

Hint: Floquet theory provides a function (the Floquet Trace $\alpha = \alpha(\mu, \epsilon)$) that characterizes linearized stability — stability if and only if $|\alpha| \leq 1$. Compute this function for $\mu$ and $\epsilon$ small.

THE END.