**18.445 HOMEWORK 4 SOLUTIONS**

**Exercise 1.** Let $X, Y$ be two random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{A} \subset \mathcal{F}$ be a sub-$\sigma$-algebra. The random variables $X$ and $Y$ are said to be independent conditionally on $\mathcal{A}$ if for every non-negative measurable functions $f, g$, we have

$$E[f(X)g(Y) | \mathcal{A}] = E[f(X) | \mathcal{A}] \times E[g(Y) | \mathcal{A}] \quad \text{a.s.}$$

Show that $X, Y$ are independent conditionally on $\mathcal{A}$ if and only if for every non-negative $\mathcal{A}$-measurable random variable $Z$, and every non-negative measurable functions $f, g$, we have

$$E[f(X)g(Y)Z] = E[f(X)Z]E[g(Y) | \mathcal{A}].$$

**Proof.** If $X$ and $Y$ are independent conditionally on $\mathcal{A}$ and $Z$ is $\mathcal{A}$-measurable, then

$$E[f(X)g(Y)Z] = E[E[f(X)g(Y)Z | \mathcal{A}]]$$

$$= E[E[f(X)g(Y) | \mathcal{A}]Z]$$

$$= E[E[f(X) | \mathcal{A}]E[g(Y) | \mathcal{A}]Z]$$

$$= E[E[f(X)]E[g(Y) | \mathcal{A}]Z | \mathcal{A}]$$

$$= E[f(X)Z]E[g(Y) | \mathcal{A}].$$

Conversely, if this equality holds for every nonnegative $\mathcal{A}$-measurable $Z$, then in particular, for every $A \in \mathcal{A}$,

$$E[f(X)g(Y) \mathbb{1}_A] = E[f(X)E[g(Y) | \mathcal{A}] \mathbb{1}_A].$$

It follows from the definition of conditional expectation that

$$E[f(X)g(Y) | \mathcal{A}] = E[f(X)E[g(Y) | \mathcal{A}] | \mathcal{A}] = E[f(Y) | \mathcal{A}]E[g(Y) | \mathcal{A}],$$

so $X$ and $Y$ are independent conditionally on $\mathcal{A}$. \hfill \Box

**Exercise 2.** Let $X = (X_n)_{n \geq 0}$ be a martingale.

(1) Suppose that $T$ is a stopping time, show that $X_T$ is also a martingale. In particular, $E[X_{T \wedge n}] = E[X_0]$.

**Proof.** Since $X$ is a martingale, first we have

$$E[||X_n^T||] \leq E[\max_{i \leq n} |X_i|] \leq \sum_{i=1}^n E[||X_i||] < \infty.$$

Moreover, for every $n \geq m$,

$$E[X_n^T | \mathcal{F}_{n-1}] = E[X_{n-1}^T + (X_n - X_{n-1}) \mathbb{1}_{T > n-1} | \mathcal{F}_{n-1}]$$

$$= E[X_{n-1}^T + \mathbb{1}_{T > n-1} E[X_n - X_{n-1} | \mathcal{F}_{n-1}]]$$

$$= E[X_{n-1}^T].$$

We conclude that $X_T$ is a martingale. \hfill \Box

(2). Suppose that \( S \leq T \) are bounded stopping times, show that \( \mathbb{E}[X_T | \mathcal{F}_S] = X_S \), a.s. In particular, \( \mathbb{E}[X_T] = \mathbb{E}[X_S] \).

**Proof.** Suppose \( S \) and \( T \) are bounded by a constant \( N \in \mathbb{N} \). For \( A \in \mathcal{F}_S \),

\[
\mathbb{E}[X_N 1_A] = \sum_{i=1}^{N} \mathbb{E}[X_N 1_A 1_{S=i}]
\]

\[
= \sum_{i=1}^{N} \mathbb{E}\left[\mathbb{E}[X_N | \mathcal{F}_S] 1_A 1_{S=i}\right]
\]

\[
= \sum_{i=1}^{N} \mathbb{E}\left[\mathbb{E}[X_N | \mathcal{F}_i] 1_A 1_{S=i}\right]
\]

\[
= \sum_{i=1}^{N} \mathbb{E}[X_i 1_A 1_{S=i}]
\]

\[
= \mathbb{E}[X_S 1_A],
\]

so \( \mathbb{E}[X_N | \mathcal{F}_S] = X_S \). Similarly, \( \mathbb{E}[X_N | \mathcal{F}_T] = X_T \). We conclude that

\[
\mathbb{E}[X_T | \mathcal{F}_S] = \mathbb{E}\left[\mathbb{E}[X_N | \mathcal{F}_T] | \mathcal{F}_S\right] = \mathbb{E}[X_N | \mathcal{F}_S] = X_S.
\]

\(\square\)

(3). Suppose that there exists an integrable random variable \( Y \) such that \( |X_n| \leq Y \) for all \( n \), and \( T \) is a stopping time which is finite a.s., show that \( \mathbb{E}[X_T] = \mathbb{E}[X_0] \).

**Proof.** Since \( |X_n| \leq Y \) for all \( n \) and \( T \) is finite a.s., \( |X_{n\wedge T}| \leq Y \). Then the dominated convergence theorem implies that

\[
\lim_{n \to \infty} \mathbb{E}[X_{n\wedge T}] = \mathbb{E}\left[\lim_{n \to \infty} X_{n\wedge T}\right] = \mathbb{E}[X_T].
\]

As \( n \wedge T \) is a bounded stopping time, Part (2) implies that \( \mathbb{E}[X_{n\wedge T}] = \mathbb{E}[X_0] \). Hence we conclude that \( \mathbb{E}[X_T] = \mathbb{E}[X_0] \). \(\square\)

(4). Suppose that \( X \) has bounded increments, i.e. \( \exists M > 0 \) such that \( |X_{n+1} - X_n| \leq M \) for all \( n \), and \( T \) is a stopping time with \( \mathbb{E}[T] < \infty \), show that \( \mathbb{E}[X_T] = \mathbb{E}[X_0] \).

**Proof.** We can write \( \mathbb{E}[X_T] = \mathbb{E}[X_0] + \mathbb{E}[\sum_{i=1}^{T} (X_i - X_{i-1})] \), so it suffices to show that the last term is zero. Note that

\[
\mathbb{E}[|\sum_{i=1}^{T} (X_i - X_{i-1})|] \leq \mathbb{E}[\sum_{i=1}^{T} |X_i - X_{i-1}|] \leq M \mathbb{E}[T] < \infty.
\]

Then the dominated convergence theorem implies that

\[
\mathbb{E}\left[\sum_{i=1}^{T} (X_i - X_{i-1})\right] = \mathbb{E}\left[\sum_{i=1}^{\infty} (X_i - X_{i-1}) 1_{T \geq i}\right]
\]

\[
= \sum_{i=1}^{\infty} \mathbb{E}[(X_i - X_{i-1}) 1_{T \geq i}]
\]

\[
= \sum_{i=1}^{\infty} \mathbb{E}[X_i - X_{i-1}] \mathbb{P}[T \geq i]
\]

\[
= 0,
\]

where we used that \( X_i - X_{i-1} \) is independent of \( \{T \geq i\} = \{T < i - 1\} \) as \( T \) is a stopping time of the martingale \( X \). \(\square\)

**Exercise 3.** Let \( X = (X_n)_{n \geq 0} \) be Gambler’s ruin with state space \( \Omega = \{0, 1, 2, ..., N\} \):

\[
X_0 = k, \quad \mathbb{P}[X_{n+1} = X_n + 1 | X_n] = \mathbb{P}[X_{n+1} = X_n - 1 | X_n] = 1/2, \quad \tau = \min\{n : X_n = 0 \text{ or } N\}.
\]
(1). Show that \( Y = (Y_n := X_n^2 - n)_{n \geq 0} \) is a martingale.

**Proof.** By the definition of \( X \),

\[
\mathbb{E}[Y_n | \mathcal{F}_{n-1}] = \mathbb{E}[X_n^2 - n | \mathcal{F}_{n-1}]
\]

\[
= \mathbb{E}[(X_n - X_{n-1})^2 + 2(X_n - X_{n-1})X_{n-1} + X_{n-1}^2 - n | \mathcal{F}_{n-1}]
\]

\[
= \mathbb{E}[(X_n - X_{n-1})^2 | X_{n-1}] + 2\mathbb{E}[X_n - X_{n-1} | X_{n-1}]X_{n-1} + X_{n-1}^2 - n
\]

\[
= 1 + 0 + X_{n-1}^2 - n = Y_{n-1},
\]

so \( Y \) is a martingale. \( \square \)

(2). Show that \( Y \) has bounded increments.

**Proof.** It is clear that

\[
|Y_n - Y_{n-1}| = |X_n^2 - X_{n-1}^2 - 1|
\]

\[
\leq |X_n + X_{n-1}| |X_n - X_{n-1}| + 1
\]

\[
\leq |X_n| + 1 + |X_{n-1}| + 1
\]

\[
\leq 2N + 2,
\]

so \( Y \) has bounded increments. \( \square \)

(3). Show that \( \mathbb{E}[\tau] < \infty \).

**Proof.** First, let \( \alpha \) be the probability that the chain increases for \( N \) consecutive steps, i.e.

\[
\alpha = \mathbb{P}[X_{i+1} - X_i = 1, X_{i+2} - X_{i+1} = 1, \ldots, X_{i+N} - X_{i+N-1} = 1]
\]

which is positive and does not depend on \( i \). If \( \tau > mN \), then the chain never increases \( N \) times consecutively in the first \( mN \) steps. In particular,

\[
\{\tau > mN\} \subset \bigcap_{i=0}^{m-1} \{X_{iN+1} - X_{iN} = 1, X_{iN+2} - X_{iN+1} = 1, \ldots, X_{iN+N} - X_{iN+N-1} = 1\}^c.
\]

Since the events on the right-hand side are independent and each have probability \( 1 - \alpha < 1 \),

\[
\mathbb{P}[\tau > mN] \leq (1 - \alpha)^m.
\]

For \( mN \leq l < (m + 1)N \), \( \mathbb{P}[\tau > l] \leq \mathbb{P}[\tau > mN] \), so

\[
\mathbb{E}[\tau] = \sum_{l=0}^{\infty} \mathbb{P}[\tau > l] \leq \sum_{m=0}^{\infty} N \mathbb{P}[\tau > mN] \leq N \sum_{m=0}^{\infty} (1 - \alpha)^m < \infty.
\]

\( \square \)

(4). Show that \( \mathbb{E}[\tau] = k(N - k) \).

**Proof.** Since \( \mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] = 0 \) and |\( X_{n+1} - X_n | = 1 \), \( X \) is a martingale with bounded increments. We also showed that \( Y \) is a martingale with bounded increments. As \( \mathbb{E}[\tau] < \infty \), Exercise 2 Part (4) implies that

\[
k = \mathbb{E}[X_0] = \mathbb{E}[X_\tau] = \mathbb{P}[X_\tau = 0] 
\cdot 0 + \mathbb{P}[X_\tau = N] \cdot N
\]

and

\[
k^2 = \mathbb{E}[Y_0] = \mathbb{E}[Y_\tau] = \mathbb{E}[X_\tau^2] - \mathbb{E}[\tau].
\]

Then (1) gives, \( \mathbb{P}[X_\tau = N] = k/N \). Hence it follows from (2) that

\[
\mathbb{E}[\tau] = \mathbb{E}[X_\tau^2] - k^2 = \mathbb{P}[X_\tau = 0] 
\cdot 0 + \mathbb{P}[X_\tau = N] \cdot N^2 - k^2 = kN \rightarrow k^2 = k(N - k).
\]

\( \square \)

**Exercise 4.** Let \( X = (X_n)_{n \geq 0} \) be the simple random walk on \( \mathbb{Z} \).
(1) Show that \(Y_n := X_n^3 - 3nX_n\) is a martingale.

Proof. We have
\[
E[Y_n - Y_{n-1} | \mathcal{F}_{n-1}] = E[X_n^3 - 3nX_n - X_{n-1}^3 + 3(n-1)X_{n-1} | \mathcal{F}_{n-1}]
\]
\[
= E[(X_n - X_{n-1})^3 + 3(X_n - X_{n-1})^2 X_{n-1} + 3(X_n - X_{n-1}) X_{n-1}^2 - 3n(X_n - X_{n-1}) - 3X_{n-1} | \mathcal{F}_{n-1}]
\]
\[
= 0 + 3X_{n-1} + 0 - 0 - 3X_{n-1} = 0,
\]
so \(Y\) is a martingale. \(\square\)

(2) Let \(\tau\) be the first time that the walker hits either 0 or \(N\). Show that, for \(0 \leq k \leq N\), we have
\[
E_k[\tau \mid X_\tau = N] = \frac{N^2 - k^2}{3}.
\]

Proof. Since \(0 \leq X_n^\tau \leq N\), the martingale \(Y^\tau\) is bounded and thus has bounded increments. The stopping time \(\tau\) is the same as in Exercise 3, so the same argument implies that
\[
k^3 = E[Y_0] = E[Y_\tau] = E[X^3_\tau] - 3E[\tau X_\tau].
\]
We compute that \(E[X^3_\tau] = P[X_\tau = 0] \cdot 0 + P[X_\tau = N] \cdot N^3 = kN^2\). Hence
\[
kN^2 - k^3 = E[\tau X_\tau] = P[X_\tau = 0] \cdot 0 + P[X_\tau = N] \cdot E[\tau N \mid X_\tau = N] = kE[\tau \mid X_\tau = N].
\]
We conclude that
\[
E[\tau \mid X_\tau = N] = \frac{N^2 - k^2}{3}.
\]
\(\square\)

Exercise 5. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with filtration \((\mathcal{F}_n)_{n \geq 0}\).

(1) For any \(m, m' \geq n\) and \(A \in \mathcal{F}_n\), show that \(T = m1_A + m'1_{A^c}\) is a stopping time.

Proof. Assume without loss of generality that \(m \leq m'\) (since we can flip the roles of \(A\) and \(A^c\)). If \(l < m\), then \(\{T \leq l\} = \emptyset \in \mathcal{F}_l\). If \(m \leq l < m'\), then \(\{T \leq l\} = A \in \mathcal{F}_n \subset \mathcal{F}_l\) as \(n \leq m \leq l\). If \(l \geq m'\), then \(\{T \leq l\} = \Omega \in \mathcal{F}_l\). Hence \(T\) is a stopping time. \(\square\)

(2) Show that an adapted process \((X_n)_{n \geq 0}\) is a martingale if and only if it is integrable, and for every bounded stopping time \(T\), we have \(E[X_T] = E[X_0]\).

Proof. The “only if” part was proved in Exercise 2 Part (2) with \(S \equiv 0\).

Conversely, suppose for every bounded stopping time \(T\), we have \(E[X_T] = E[X_0]\). In particular, \(E[X_m] = E[X_0]\) for every \(m \in \mathbb{N}\). Moreover, for \(n \leq m\) and \(A \in \mathcal{F}_n\), Part (1) implies that \(T = n1_A + m1_{A^c}\) is a bounded stopping time. Thus
\[
E[X_m] = E[X_0] = E[X_T] = E[X_n1_A + X_m1_{A^c}],
\]
so \(E[X_m1_A] = E[X_n1_A]\). By definition, this means \(E[X_m \mid \mathcal{F}_n] = X_n\), so \(X\) is a martingale. \(\square\)

Exercise 6. Let \(X = (X_n)_{n \geq 0}\) be a martingale in \(L^2\).
(1). Show that its increments \((X_{n+1} - X_n)_{n \geq 0}\) are pairwise orthogonal, i.e. for all \(n \neq m\), we have \(\mathbb{E}[(X_{n+1} - X_n)(X_{m+1} - X_m)] = 0\).

Proof. First, note that for any \(n \leq m\),
\[
\mathbb{E}[X_n X_m] = \mathbb{E}[\mathbb{E}[X_n X_m | \mathcal{F}_n]] = \mathbb{E}[X_n \mathbb{E}[X_m | \mathcal{F}_n]] = \mathbb{E}[X_n^2].
\]
Now assume without loss of generality that \(n < m\). Then
\[
\mathbb{E}[(X_{n+1} - X_n)(X_{m+1} - X_m)] = \mathbb{E}[X_{n+1}X_{m+1}] - \mathbb{E}[X_nX_{m+1}] - \mathbb{E}[X_{n+1}X_m] + \mathbb{E}[X_nX_m]
= \mathbb{E}[X_{n+1}^2] - \mathbb{E}[X_n^2] - \mathbb{E}[X_{n+1}^2] + \mathbb{E}[X_n^2] = 0.
\]
\(\square\)

(2). Show that \(X\) is bounded in \(L^2\) if and only if
\[
\sum_{n \geq 0} \mathbb{E}[(X_{n+1} - X_n)^2] < \infty.
\]

Proof. Note that
\[
\mathbb{E}[X_0(X_{n+1} - X_n)] = \mathbb{E}[X_0^2] - \mathbb{E}[X_0^2] = 0
\]
by the computation in Part (1). Thus for any \(m\), we have
\[
\mathbb{E}[X_m^2] = \mathbb{E}\left[(X_0 + \sum_{n=0}^{m-1} (X_{n+1} - X_n))^2\right] = \mathbb{E}[X_0^2] + \sum_{n=0}^{m-1} \mathbb{E}[(X_{n+1} - X_n)^2]
\]
where the cross terms disappear by Part (1). Therefore,
\[
\sup_{m \geq 0} \mathbb{E}[X_m^2] = \mathbb{E}[X_0^2] + \sum_{n \geq 0} \mathbb{E}[(X_{n+1} - X_n)^2].
\tag{3}
\]

If \(X\) is bounded in \(L^2\), i.e. the left-hand side in (3) is bounded, then the sum on the right-hand side is bounded. Conversely, if the sum is bounded, since \(X_0\) is in \(L^2\), the left-hand side is also bounded. \(\square\)
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