18.445 Introduction to Stochastic Processes
Lecture 10: Hitting times

Hao Wu
MIT

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Recall

- Consider a network \( (G = (V, E), \{c(e) : e \in E\}) \). The effective resistance is defined by

\[
R(a \leftrightarrow z) = \frac{W(a) - W(z)}{||l||}.
\]

- Consider a random walk on the network, the Green’s function is defined by

\[
G_\tau(a, x) = \mathbb{E}[\# \text{visits to } x \text{ before } \tau].
\]

- We have that

\[
G_{\tau z}(a, a) = c(a)R(a \leftrightarrow z).
\]

Today’s Goal

- hitting time
- commute time
- transitive network
Target time

Suppose that $(X_n)_{n \geq 0}$ is an irreducible Markov chain with transition matrix $P$ and stationary measure $\pi$. Let $\tau_x$ be the hitting time:

$$\tau_x = \min\{n \geq 0 : X_n = x\}.$$

**Lemma**

The quantity

$$\sum_x E_a[\tau_x] \pi(x)$$

does not depend on $a$; and we call it target time and denote it by $t_\circ$. 
Hitting time

Definition

\[ t_{hit} := \max_{x,y} E_x[\tau_y] \geq t_{\odot}. \]

Lemma

Suppose that the chain is irreducible with stationary measure \( \pi \). Then

\[ t_{hit} \leq 2 \max_w E_{\pi}[\tau_w]. \]

Theorem

For an irreducible transitive Markov chain, we have

\[ t_{hit} \leq 2t_{\odot}. \]
Transitive Markov chain

Roughly, a transitive Markov chain “looks the same” from any point in the state space.

Definition
A Markov chain is called transitive if for each pair \((x, y) \in \Omega \times \Omega\), there is a bijection \(\varphi : \Omega \to \Omega\) such that

\[
\varphi(x) = y; \quad P(\varphi(z), \varphi(w)) = P(z, w), \forall z, w.
\]

Example: simple random walk on \(N\)-cycle, on hypercube.

Lemma
For a transitive Markov chain on finite state space \(\Omega\), the uniform measure is stationary.
Commute time

Definition
Suppose that the Markov chain starts from $X_0 = a$. The commute time between $a$ and $b$ is defined by

$$
\tau_{ba} = \min\{n \geq \tau_b : X_n = a\}.
$$

Theorem (Commute Time Identity)
Consider a random walk on the network $(G = (V, E), \{c(e) : e \in E\})$, we have

$$
\mathbb{E}_a[\tau_{ba}] = \mathbb{E}_a[\tau_b] + \mathbb{E}_b[\tau_a] = c_G R(a \leftrightarrow b).
$$

Lemma
Suppose that the Markov chain is irreducible with stationary measure $\pi$. Suppose that $\tau$ is a stopping time satisfying $\mathbb{P}_a[X_{\tau} = a] = 1$. Then

$$
G_\tau(a, x) = \mathbb{E}_a[\tau] \pi(x).
$$
Transitive network

Generally, $E_a[\tau_b]$ and $E_b[\tau_a]$ can be very different (see Exercise 10.3). However, if the network is transitive, they are equal.

Definition

A network $(G = (V, E), \{c(e) : e \in E\})$ is transitive if for each pair $(x, y) \in V \times V$, there exists a bijection $\varphi : V \to V$ such that

$$\varphi(x) = y; \quad c(\varphi(z), \varphi(w)) = c(z, w), \forall z, w.$$

Remark: The random walk on a transitive network is a transitive Markov chain.

Theorem

For the random walk on a transitive (connected) network, for any vertices $a$ and $b$, we have

$$E_a[\tau_b] = E_b[\tau_a].$$
Summary

For random walk on network
- \( t_{\odot} \leq t_{hit} \leq 2 \max_w \mathbb{E}_{\pi}[\tau_w]. \)
- \( \mathbb{E}_a[\tau_{ba}] = c_G R(a \leftrightarrow b). \)

For random walk on transitive network
- \( t_{\odot} \leq t_{hit} \leq 2t_{\odot}. \)
- \( \mathbb{E}_a[\tau_b] = \mathbb{E}_b[\tau_a]. \)
- \( 2\mathbb{E}_a[\tau_b] = c_G R(a \leftrightarrow b). \)
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