1. Let \( f_0(x) = 1 - |x| \) for \(|x| \leq 1\) and 0 elsewhere. Let \( f_\theta(x) = f_0(x - \theta) \). If we have i.i.d. observations \( X_1, ..., X_n \) with density \( f_\theta \), we can estimate \( \theta \) by (a) the sample mean \( \bar{X} \), (b) the sample median \( m_n \), or (c) the Hodges-Lehmann estimator \( \hat{\theta}_{HL} \). For each estimator \( T = T_n \), find the limiting distribution of \( \sqrt{n}(T_n - \theta) \). A smaller variance means the estimator is more efficient. Rank the estimators in order of efficiency in this case.

2. For \( f_0 \) a Cauchy distribution, \( \bar{X} \) no longer converges to \( \theta \), so one might say that the asymptotic variance is infinite. Proceed as in problem 1 for the other two estimators and compare them for efficiency.

3. Randles and Wolfe, problem 11.5.4, but omit the distribution-free question, just find the null distribution.

4. Let \( F_m \) be an empirical distribution function for \( F \) and independent of it let \( G_n \) be an empirical distribution function for \( G \), each based on i.i.d. observations. Let

\[
\zeta_{m,n}(x) := \sqrt{\frac{mn}{m+n}}(F_m - G_n)(x).
\]

Under the null hypothesis \( H_0 : F = G \), \( \zeta_{m,n} \) converges as \( m, n \to \infty \) in distribution to \( y_F(x) \) where \( y_t, \ 0 \leq t \leq 1 \) is a Brownian bridge process. Let \( KS_{m,n} \) be the Kolmogorov-Smirnov statistic

\[
KS_{m,n} := \sup_x |\zeta_{m,n}(x)|.
\]

If \( F = G \) is continuous then for large \( m, n \), this has asymptotically the distribution of \( \sup_{0 < t < 1} |y_t| \) given by RAP, Proposition 12.3.4. Assuming that the asymptotic distribution is valid, would \( H_0 \) be rejected at the \( \alpha = 0.05 \) level if \( KS_{m,n} = 1.5? \)

5. Find an \( n_0 \) such that for \( m, n \geq n_0 \) it can be proved from the Bretagnolle-Massart theorem 1.1 that \( P_0(KS_{m,n} \geq 2) \leq 0.05 \). Hint: In the Bretagnolle-Massart handout, make the right side of (1.2) \( \leq 0.005 \). This determines \( x \). Find \( \eta \) such that \( P(\sup_t |y_t| \geq \eta) \leq 0.04 \) and find \( n \) large enough so that \( (x + c \cdot \log n)/\sqrt{n} \leq (2 - \eta)/2 \) Try \( n_0 = v \cdot 10^5 \) for some single-digit integer \( v \).