Plane Crystallographic Groups with Point Group $D_1$.

This note describes discrete subgroups $G$ of isometries of the plane $P$ whose translation lattice $L$ contains two independent vectors, and whose point group $\overline{G}$ is the dihedral group $D_1$, which consists of the identity and a reflection about the origin. Among the ten possible point groups $C_n$ or $D_n$ with $n = 1, 2, 3, 4, 6$, the analysis of $D_1$ is among the most complicated. There are three different types of group with this point group.

Let $G$ be a group of the type that we are considering. We choose coordinates so that the reflection in $\overline{G}$ is about the horizontal axis. As in the text, we put bars over symbols that represent elements of the point group $\overline{G}$ to avoid confusing them with the elements of $G$. So we denote the reflection in $\overline{G}$ by $\overline{v}$.

The lattice $L$ consists of the vectors $v$ such that $t_v$ is in $G$, and we know that elements of $\overline{G}$ map $L$ to $L$. If $v$ is in $L$, $\overline{v}$ is also in $L$.

I. The shape of the lattice

Proposition 1. There are horizontal and vertical vectors $a = (a_1, 0)^t$ and $b = (0, b_2)^t$ respectively, such that, with $c = \frac{1}{2}(a + b)$, $L$ is one of the two lattices $L_1$ or $L_2$, where

\[
L_1 = Za + Zb, \quad \text{is a 'rectangular' lattice, and} \\
L_2 = Za + Zc, \quad \text{is a 'triangular' lattice.}
\]

Since $b = 2c - a$, $L_1 \subset L_2$.

The lattice $L_1$ is called ‘rectangular’ because the horizontal and vertical lines through its points divide the plane into rectangles. The lattice $L_2$ is obtained by adding to $L_1$ the midpoints of every one of these rectangles. There are two scale parameters in the description of $L$ — the lengths of the vectors $a$ and $b$. The usual classification of discrete groups disregards these parameters, but the rectangular and isosceles lattices are considered different.

Proof of the proposition. Let $v = (v_1, v_2)^t$ be an element $L$ not on either coordinate axis. Then $\overline{v}v = (v_1, -v_2)^t$ is in $L$. So are the vectors $v + \overline{v}v = (2v_1, 0)^t$, and $v - \overline{v}v = (0, 2v_2)^t$. These are nonzero horizontal and vertical vectors in $L$, respectively.

We choose $a_1$ to be the smallest positive real number such that $a = (a_1, 0)^t$ is in $L$. This is possible because $L$ contains a nonzero horizontal vector and it is a discrete group. Then the horizontal vectors in $L$ will be integer multiples of $a$. We choose $b_2$ similarly, so that the vertical vectors in $L$ are the integer multiples of $b = (0, b_2)^t$, and we let $L_1$ be the rectangular lattice $Za + Zb = \{am + bn \mid m, n \in \mathbb{Z}\}$. Then $L_1 \subset L$.

Suppose that $L \neq L_1$. We choose a vector $v = (v_1, v_2)^t$ in $L$ and not in $L_1$. It will be a linear combination of the independent vectors $a$ and $b$, say $v = ax + by = (a_1x, b_2y)^t$, with $x, y \in \mathbb{R}$. We write $x = m + x_0$ with $m \in \mathbb{Z}$ and $0 \leq x_0 < 1$, and we write $y = bn + y_0$ in the analogous way. Then $v = (am + bn) + (ax_0 + by_0)$. The vector $am + bn$ is in $L_1$. We subtract this vector from $v$, and are reduced to the case that $v = ax + by$, with $0 \leq x, y < 1$. As we saw above, $(2v_1, 0)^t$ is in $L$. Since this is a horizontal vector, $2v_1$ is an integer multiple of $a_1$, and since $0 \leq v_1 < a_1$, there are only two possibilities: $v_1 = 0$ or $\frac{1}{2}a_1$. Similarly, $v_2 = 0$ or $v_2 = \frac{1}{2}b_2$. Thus $v$ is one of the four vectors $0, \frac{1}{2}a_1, \frac{1}{2}b, c$. It is not 0 because $v \notin L_1$. It is not $\frac{1}{2}a_1$ because $a$ is a horizontal vector of minimal length in $L$, and it is not $\frac{1}{2}b$ because $b$ is a vertical vector of minimal length. Thus $v = c$, and $L = L_2$. \qed
II. The glides in $G$.

We recall that the homomorphism $\pi : M \to O_2$ sends an isometry $m = t_{v}\varphi$ to the orthogonal operator $\varphi$. The point group $\Gamma$ is the image of $G$ in $O_2$. So there is an element $g$ in $G$ such that $\pi(g) = \tau$, and $g = t_{u}\tau r$ for some vector $u = (u_1, u_2)^t$. It is important to keep this in mind: Though $t_{u}\tau r$ is in $G$, the translation $t_{u}$ by itself needn’t be in $G$.

**Lemma 2.** Let $H$ be the subgroup of translations $t_{v}$ in $G$. So $L = \{v \mid t_{v} \in H\}$, and $H = \{t_{v} \mid v \in L\}$.

(i) $G$ is the union of two cosets $H \cup Hg$, where $g$ can be any element not in $H$.

(ii) $g$ has the form $t_{u}\tau r$, where $u + \tau r$ is in the subgroup $a\mathbb{Z}$.

(iii) Let $L$ be a lattice of the form $L_1$ or $L_2$, and let $H = \{t_{v} \mid v \in L\}$. Let $u$ be a vector such that $u + \tau r u$ is in $a\mathbb{Z}$. Then the set $H \cup Hg$ is a discrete subgroup of $M$.

**proof.** (i) Let $\pi_G$ denote the restriction of $\pi$ to $G$. The kernel of this homomorphism is the group $H$, and its image $\Gamma$ contains two elements. Therefore there are two cosets of $H$ in $G$.

(ii) We compute, using the formula $rt_{u} = t_{\tau u} r$:

$$g^2 = t_{u} r t_{u} r = t_{u + \tau u} r^2 = t_{u + \tau u}$$

This is an element of $G$, so $u + \tau u$ is a horizontal vector in $L$, an integer multiple of $a$.

The verification of (iii) is similar to the computation made in (ii), and we omit it. □

We check that the isometry $g = t_{u}\tau r$ is a reflection or a glide with horizontal glide line $\ell$ defined by $x_2 = \frac{1}{2}u_2$:

$$t_{u}\tau r \left( \begin{array}{c} x_1 \\ \frac{1}{2}u_2 \end{array} \right) = t_{u} \left( \begin{array}{c} x_1 \\ -\frac{1}{2}u_2 \end{array} \right) = \left( \begin{array}{c} x_1 + u_1 \\ \frac{1}{2}u_2 \end{array} \right)$$

So $g$ is a horizontal glide along $\ell$, as asserted. The glide vector is $(u_1, 0)^t$.

Since the glide line $\ell$ is horizontal, we can shift coordinates to make it the horizontal axis. This changes the vector $u$, which becomes the horizontal vector $(u_1, 0)^t$. Then $\tau u = u$, and therefore $g^2 = t_{2u}$ (see (3)).

So $t_{2u}$ is in $G$, and $2u$ is in $L$. Since $2u$ is a horizontal vector, it is an integer multiple of $a$. We adjust $u$, multiplying $g$ on the left by a power of $t_{a}$ to make $u = 0$ or $\frac{1}{2}a$.

The two dichotomies

$$L = L_1 \text{ or } L_2,$$

and $u = 0$ or $\frac{1}{2}a$,

leave us with four possibilities.

To complete the discussion we must decide whether or not such groups exist, and whether they are different. They do exist, because $H \cup Hg$ is a group (Lemma 2 (iii)). And the two types of lattice are considered different. But when the element $g$ we have found is a glide, $G$ might still contain a reflection. This happens when $L = L_2$ and $u = \frac{1}{2}a$. In that case, $c = \frac{1}{2}(a + b)$ is in $L$, and so $t_{-c}g = t_{-\frac{a+b}{2}}r$ is an element of $G$.

Because $-\frac{1}{2}b$ is a vertical vector, this motion is a reflection (about the horizontal line $x_2 = \frac{1}{2}b$). Shifting coordinates once more eliminates this case. This phenomenon doesn’t happen when $L = L_1$, so we are left with three types of group.

**Theorem.** Let $G$ be a discrete group of isometries of the plane whose point group is the dihedral group $D_1 = \{1, \tau\}$. Let $H = \{t_{v} \in G\}$ be its subgroup of translations.

(i) The lattice $L = \{v \mid t_{v} \in G\}$ has one of the forms $L_1$ or $L_2$ given in Proposition 1.

(ii) Let $u = \frac{1}{2}a$ and let $g = t_{u}\tau r$. Coordinates in the plane can be chosen so that,

a) if $L = L_1$, $G = H \cup Hr$ or $G = H \cup Hg$, and

b) if $L = L_2$, $G = H \cup Hr$. □