18.725: EXERCISE SET 8

DUE THURSDAY NOVEMBER 6

(1) Assume the characteristic of \( k \) is not 2. Let \( b : X \to \mathbb{A}^2 \) be the blow-up of \( \mathbb{A}^2 \) at the origin \((0,0)\), and let \( U = \mathbb{A}^2 - \{(0,0)\} \).

(i) Show that the morphism \( b^{-1}(U) \to U \) is an isomorphism.

(ii) If \( Z \subset \mathbb{A}^2 \) is a closed subset not equal to \( \{(0,0)\} \), the strict transform of \( Z \) is defined to be the closure of \( Z \cap U \) in \( X \), where \( Z \cap U \) is viewed as a subset of \( X \) via the isomorphism in (i). Compute the strict transform of \( V(xy) \subset \mathbb{A}^2 \).

(iii) Compute the strict transform of \( V(y^2 - x^2(x + 1)) \subset \mathbb{A}^2 \).

(2) Let \( X \) be a variety and \( x \in X \) a point. Denote by \( \mathfrak{m}_x \subset O_{X,x} \) the maximal ideal of the local ring \( O_{X,x} \). Since the residue field of \( O_{X,x} \) is \( k \), the quotient \( \mathfrak{m}/\mathfrak{m}^2 \) is a \( k \)-vector space. Show that

\[
\dim_k(\mathfrak{m}/\mathfrak{m}^2) \geq \dim(X).
\]

Show that if \( O_{X,x} \) is a regular local ring, then this in fact is an equality.

(3) Let \( f : X \to Y \) be a non-constant finite morphism of varieties with \( \dim(X) = \dim(Y) = 1 \). Fix a point \( y \in Y \) and let \( x_1, \ldots, x_r \) be the points in \( f^{-1}(y) \). Assume that the local rings \( O_{Y,y} \) and \( \{O_{X,x_i}\}_{i=1}^r \) are all regular. For each \( x_i \), let \( e(x_i) \) denote the dimension of the \( k \)-vector space \( O_{X,x_i}/\mathfrak{m}_yO_{X,x_i} \). Show that

\[
[k(X) : k(Y)] = \sum_{i=1}^r e(x_i).
\]

(4) A morphism of varieties \( f : X \to Y \) is said to be projective if for some \( n \) there is a factorization of \( f \)

\[
X \xrightarrow{j} \mathbb{P}^n \times Y \xrightarrow{p_2} Y,
\]

where \( j \) identifies \( X \) with an irreducible closed subvariety of \( \mathbb{P}^n \times Y \).

(i) Show that a projective morphism is closed.

(ii) If \( Y \) is affine, show that any finite morphism \( f : X \to Y \) is projective.

(5) Suppose the characteristic of \( k \) is \( p > 0 \). On an earlier homework, we saw an example of a morphism of varieties \( f : X \to Y \) which was not an isomorphism but was a homeomorphism on the underlying topological spaces.

(i) Let \( Y \) be an affine variety, and let \( f : X \to Y \) be a morphism of varieties whose map on underlying spaces is a homeomorphism. Show that \( X \) is affine.

(ii) Let \( R \) be the coordinate ring of \( Y \), and let \( X \) be a second affine variety with coordinate ring \( S \). Let \( f : X \to Y \) be a morphism associated to a map of rings \( f^* : R \to S \). Give

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necessary and sufficient conditions on the map $f^*$ for the morphism $f$ to be a homeomorphism on the underlying topological spaces.

(6) Let $X \subset \mathbb{A}^3$ be the zero locus of $z^2 - xy$.

(i) Show that $\dim(X) = 2$.

(ii) Find a closed subvariety $W \subset X$ of codimension 1 which is not of the form $V(g)$ for some $g \in \Gamma(X, \mathcal{O}_X)$.

(7) Fix positive integers $N$ and $r$, and let

$$F : \text{varieties} \longrightarrow \text{Set}$$

be the contravariant functor which to any variety $Y$ associates the set of polynomials in $\Gamma(Y, \mathcal{O}_Y)[X_1, \ldots, X_r]$ of degree $N$. If $g : W \to Y$ is a morphism of varieties, then the map $F(Y) \to F(W)$ is the one induced by the map

$$\Gamma(Y, \mathcal{O}_Y)[X_1, \ldots, X_r] \longrightarrow \Gamma(W, \mathcal{O}_W)[X_1, \ldots, X_r]$$

induced by $g^*$. Show that $F$ is representable. In other words, there exists a variety $M$ and an isomorphism of functors $h_M \simeq F$, where $h_M$ is the functor sending $Y$ to $\text{Hom}(Y, M)$. 