(1) Show that a quasicoherent sheaf on a quasi-projective variety X is a union of its coherent subsheaves. [Hint: reduce to the case when X is projective by replacing your sheaf by its direct image under an appropriate open embedding. If F is a quasi-coherent sheaf on X ⊂ P^n, show that every section of F|_{A^n} → X extends to a map O(−d) → F for some d. Now consider the images of the direct sum of several such maps.]

(2) Recall that the arithmetic genus of a complete curve X is the dimension of the space H^1(O_X).

Suppose that each component of X is isomorphic to P^1, two components intersect by at most one point and each such intersection is a nodal singularity (i.e., its completed local ring is isomorphic to k[[x, y]]/(xy)).

Let Γ be a graph whose vertices are indexed by components of X and two vertices are connected by an edge when the corresponding components intersect. Show that p_n(X) = 1 − χ(Γ), where p_n denotes the arithmetic genus and χ is the Euler characteristic.

(3) Let X = Spec(A) be a normal affine irreducible variety with only one regular point x ∈ X. Show that the following three statements are equivalent:
(a) Cl(X) = 0, where Cl is the divisor class group, i.e., the quotient of the group of Weil divisors by the subgroup of principal divisors.
(b) Pic(X \ x) = 0.
(c) A is UFD.

(4) Let X be as in problem 3 and let π : Y → X be a resolution of singularities of X, suppose that π^−1(X \ x) maps isomorphically to X \ x. Suppose also that the canonical line bundle K_Y is trivial and that π^−1(x) is a curve of the type described in problem 2, let D_1, . . . , D_n be the components of π^−1(x). We get a homomorphism Pic(Y) → Z^n, L → (d_i), where the restriction of L to D_i is isomorphic to O_Y(d_i). Compute the image of (the class of) O(D_i) under that homomorphism.

(5) Let G be a finite subgroup in SL(2, C) and X = C^2/G, let x ∈ X be the image of 0. It can be shown that X is normal and there exists a unique resolution Y → X satisfying the assumptions of problem 4. Moreover, the map Pic(Y) → Z^n described in problem 4 is an isomorphism. Deduce that C[x, y]^G is a UFD iff the Cartan matrix constructed from the graph Γ has determinant ±1 (in fact this determinant is always positive, so the option for it to equal −1 is not realized). Here Cartan matrix C = C_G is given by:

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1This is in fact true for not necessarily quasi-projective varieties, and even more generally, see e.g. Exercise II.5.15, in Harthshorne.
2We have only discussed how to associate a Weil divisor to a rational function in the cases when X is a curve or when X is smooth. In this problem you only need to use that such a construction exists for normal irreducible varieties and that it is compatible with restriction to an open subset.
\[ C_{ij} = 2, \quad C_{ij} = -1 \text{ if } i \neq j \text{ are connected by an edge in the graph } \Gamma \text{ and } \ C_{ij} = 0 \text { otherwise.} \]

In fact, the graph \( \Gamma \) is necessarily one of the simply-laced Dynkin graphs appearing in the classification of compact connected simple groups. The only such graph for which \( \det(C_{\Gamma}) = 1 \) (this condition is equivalent to the corresponding simple Lie group being simply-connected) corresponds to the largest simple connected compact Lie group \( E_8 \). The group \( G \) in this case is the binary icosahedral group, i.e. the preimage in the special unitary group \( SU(2) \) of the group of symmetries of a regular icosahedron under the homomorphism \( SU(2) \to PSU(2) \cong SO(3) \). The surface \( X \) is isomorphic to the surface in \( \mathbb{A}^3 \) given by the equation \( x^2 + y^3 + z^5 = 0 \), as described by Felix Klein in his book “Lectures on the icosahedron and solution of the fifth degree equations” (1884); the resolution \( Y \) can be obtained from \( X \) by 8 blow-ups, cf. Exercise V.5.8 in Hartshorne.