18.727 Topics in Algebraic Geometry: Algebraic Surfaces
Spring 2008

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1. K3 Surfaces (contd.)

Remark. Note that K3 surfaces can only be elliptic over \(\mathbb{P}^1\): on a K3 surface, however, one can have many different elliptic fibrations, though not every K3 surface has one.

2. Enriques Surfaces

Recall that such surfaces have \(\kappa(X) = 0\), \(K_X \equiv 0\), \(b_2 = 10\), \(b_1 = 0\), \(\chi(O_X) = 1\). A classical Enriques surface has \(p_g = 0\), \(q = 0\), \(\Delta = 0\), while a non-classical Enriques surface has \(p_g = 1\), \(q = 1\), \(\Delta = 2\) (which can only happen in characteristic 2). We will discuss only classical Enriques surfaces.

\textbf{Proposition 1.} For an Enriques surface, \(\omega_X \not\approx O_X\), but \(\omega_X^2 \approx O_X\).

\textit{Proof.} Since \(p_g = 0\), \(\omega_X \not\approx O_X\). By Riemann-Roch, \(\chi(O_X(-K)) = \chi(O_X) + \frac{1}{2}(-K)(-2K) = \chi(O_X) = 1\), so \(h^0(O_X(-K)) + h^0(O_X(2K)) \geq 1\). Since \(K_X \not\approx O_X\), \(K_X \not\approx O_X\), \(h^0(O_X(-K)) = 0\) (since \(-K \equiv 0\)), and so \(h^0(O_X(2K)) \geq 1\). Since \(2K \equiv 0\), \(2K = 0\), i.e. \(\omega_X^2 \approx O_X\). So the order of \(K\) in \(\text{Pic}(X)\) is 2. Note that \(\text{Pic}(X) = \text{NS}(X)\), because \(\text{Pic}^0(X) = 0\) since \(q = 0\), \(\Delta = 0\) for classical Enriques surfaces.\(\square\)

\textbf{Proposition 2.} \(\text{Pic}^\tau(X) = \mathbb{Z}/2\mathbb{Z}\), where the former object is the space of divisors numerically equivalent to zero modulo linear (or algebraic) equivalence, or similarly the torsion part of \(\text{NS}\).

\textit{Proof.} Let \(L \equiv 0\). By Riemann-Roch, \(\chi(L) = \chi(O_X) + \frac{1}{2}L \cdot (L - K) = \chi(O_X) = 1\). Thus, \(h^0(L) \neq 0\) or \(h^2(L) = h^0(K - L) \neq 0\). But both \(L\) and \(K - L\) are \(\equiv 0\), so either \(L \cong O_X\) or \(\omega \otimes L^{-1} \cong O_X\), i.e. \(L \cong \omega\).\(\square\)

\textbf{Proposition 3.} Let \(X\) be an Enriques surface. Suppose \(\text{char}(k) \neq 2\). Then \(\exists\) an \(\text{étale}\) covering \(X'\) of degree 2 of \(X\) which is a K3 surface, and the fundamental group of \(X'/X\) is \(\mathbb{Z}/2\mathbb{Z}\).

\textit{Proof.} \(K_X\) is a 2-torsion divisor class. Let \((f_{ij}) \in Z^1(\{U_i\}, O_X^*)\) be a cocycle representing \(K\). in \(\text{Pic}(X) = H^1(X, O_X^*)\). Since \(2K \sim 0\), \((f_{ij})\) is a coboundary,
so we can write is as $f_{ij}^2 = \frac{\alpha_i}{\alpha_j}$ on $U_i \cap U_j, g_i \in \Gamma(U_i, \mathcal{O}_X)$. Now $\pi : X' \to X$ defined locally by $z_i^2 = g_i$ on $U_i$ given by $\frac{z_i}{z_j} = f_{ij}$. This is étale since $\text{char}(k) \neq 2$. $\omega_{X'} = \pi^*(\omega_X) \implies \kappa(X') = 0$ as well. Since $\chi(O_{X'}) = 2\chi(O_X) = 2$, $X'$ is a K3 surface from the classification theorem. 

Remark. Over $\mathbb{C}$, in terms of line bundles, take $X' = \{ s \in L | \alpha(S^{\otimes 2}) = 1 \}$, where $\omega_X = L = \mathcal{O}(K)$ is a line bundle equipped with an isomorphism $\alpha : L^{\otimes 2} \cong \mathcal{O}_X$. The map $L \supset X' \ni s \to (x, x) \in X' \times X$ defines a nowhere vanishing section of $\pi^*L$ which is trivial, implying that $\pi^*L = K_{X'}$ is trivial. This implies that $\chi(O_{X'}) = 2$, and thus $X'$ is K3.

**Proposition 4.** Let $X'$ be a K3 surface and $i$ a fixed-point-free involution s.t. it gives rise to an étale connected covering $X' \to X$. If $\text{char}(K) \neq 2$, then $X$ is an Enriques surface.

**Proof.** $\omega_{X'} = \pi^*(\omega_X)$, and since $\omega_{X'} \equiv \mathcal{O}_{X'}, \omega_X \equiv 0, \kappa(X) = 0$, and $\chi(O_X) = \frac{1}{2}\chi(O_{X'}) = 1$. By classification, $X$ is an Enriques surface. 

Thus, Enriques surfaces are quotients of K3 surfaces by fixed-point free involutions.

**Example.** The smooth complete intersection of 3 quadrics in $\mathbb{P}^5$ is a K3 surface. Let $f_i = Q_i(x_0, x_1, x_2) + Q'_i(x_3, x_4, x_5)$ for $i = 1, 2, 3$, where $Q_i, Q'_i$ are homogeneous quadratic forms; the $f_i$ cut out $X'$, a K3 surface. Now, let $\sigma : \mathbb{P}^5 \to \mathbb{P}^5, \sigma(x_0 : \cdots : x_5) = (x_0 : x_1 : x_2 : -x_3 : -x_4 : -x_5)$ be an involution. Note that $\sigma(X') = X'$. Generically, the 3 conics $Q_i = 0$ in $\mathbb{P}^2$ (respectively the conics $Q'_i = 0$) have no points in common, implying that $\sigma' = \sigma_{X'}$ has no fixed points in $X'$, giving us an Enriques surface as above.

**Theorem 1.** Every Enriques surface is elliptic (or quasielliptic).

**Proof.** Exercise. 

3. Bielliptic surfaces

This is the fourth class of surfaces with $\kappa(X) = 0 : b_2 = 2, \chi(O_X) = 0, b_1 = 2, K_X \equiv 0$. There are two cases:

1. $p_g = 0, q = 1, \Delta = 0$: the classical, bielliptic/hyperelliptic surface.
2. $p_g = 1, q = 2, \Delta = 2$, which only happens in positive characteristic.

In either case, $b_1 = 2 \implies s = b_2^2 = 1 = \dim \text{Alb}(X)$, so the Albanese variety is an elliptic curve.

**Theorem 2.** The map $f : X \to \text{Alb}(X)$ has all fibers either smooth elliptic curves, or all rational curves, each having one singular point which is an ordinary cusp. The latter case happens only in characteristic 2 or 3.
**Proof.** Let $B = \text{Alb}(X)$, $b \in B$ a closed point, $F = F_b = f^{-1}(b)$. Then $F^2 = 0, F \cdot K = 0 \implies p_a(f) = 1 \implies f : X \to B$ is an elliptic or quasi-elliptic fibration (the latter only in characteristic 2 or 3). All the fibers of $f$ are irreducible (if we had a reducible fiber $F = \sum a_iE_i$, then the classes of $F, E_i$, and $H$ (the hyperplane section) would give 3 independent classes in $\text{NS}(X)$, implying that $b_2 \geq \rho \geq 3$ by the Igusa-Severi inequality, a contradiction). Similarly, one can show that there are no multiple fibers, implying that all fibers are integral. If the general fiber is smooth (or any closed fiber is smooth), then $f^*\omega, \omega \in F^0(B, \omega_B)$ is a regular 1-form on $X$, vanishes exactly where $f$ is not smooth, implying that it is a global section of $\Omega^1_{X/k}$ whose zero locus is either empty or of pure codimension 2. A result of Grothendieck shows that the degree of the zero locus is $c_2(\Omega^1_{X/k}) = c_2 = 2 - 2b_1 + b_2 = 0$, implying that $f^*\omega$ is everywhere nonzero and $f$ is smooth. 

**Remark.** If all fibers of the Albanese map are smooth, call it a hyperelliptic/bielliptic surface. If all fibers of the Albanese map are singular, call it a quasihyperelliptic/quasibielliptic surface.

Next, we find a second elliptic fibration.

**Theorem 3.** Let $X$ be as above, $f : X \to B = \text{Alb}(X)$ a hyperelliptic or quasihyperelliptic fibration. Then $\exists$ another elliptic fibration $g : X \to \mathbb{P}^1$.

**Proof.** (Idea) Find an indecomposable curve $C$ of canonical type s.t. $C \cdot F_t > 0$ for all $t \in B$, where $F_t = f^*(t)$. First note the following.

**Definition 1.** Let $X$ be a minimal surface and $D = \sum n_i E_i > 0$ be an effective divisor on $X$. We say that $D$ is a divisor (or curve) of canonical type if $K \cdot E_i = D \cdot E_i = 0$ for all $i = 1, \cdots, r$. If $D$ is also connected, and the g.c.d. of the integers $n_i$ is 1, then we say that $D$ is an indecomposable divisor (or curve) of canonical type.

**Theorem 4.** Let $X$ be a minimal surface with $K^2 = 0$ and $K \cdot C \geq 0$ for all curves $C$ on $X$. If $D$ is an indecomposable curve of canonical type on $X$, then $\exists$ an elliptic or quasi-elliptic fibration $f : X \to B$ obtained from the Stein factorization of the morphism $\phi_{[nD]} : X \to \mathbb{P}(H^0(\mathcal{O}_X(nD))^\vee)$ (dual, since the points of $x$ are functionals on $H^0(\mathcal{O}_X(nD))$) for some $n > 0$.

We will prove this later, and for now, we return to the proof for hyperelliptic surfaces. If we can find such a $C$ of canonical type, then we get an elliptic or quasielliptic fibration $g : X \to B'$ s.t. $(F_t, G_{t'}) > 0$ for all $t \in B, t' \in B'$, where $G_{t'} = g^{-1}(t')$. If $g$ where quasielliptic, then the general fiber $G_t$ would be a rational curve, implying that $f(G_{t'})$ is a point (since $B$ is an elliptic curve) and $G_{t'} \subset F_t$ for some $t$, contradicting $(F_t, G_{t'}) > 0$. So $g$ is in fact an elliptic fibration. Similarly, it is not hard to see that the base must be $\mathbb{P}^1$. How do we find $C$?
Let $H$ be a hyperplane section, $F_0$ a fiber of $f$. Let $D = aH + bF_0$ so that $D^2 = 0$, $D \cdot F > 0$ (e.g. $b = -H^2, a = 2(H \cdot F_0)$). Then one can prove that, for some $t \in B, D_t = D + F_t - F_0$ has $|D_t| \neq \emptyset$.

Now we have two different elliptic fibrations “transversal” to each other.

**Theorem 5.** Let $X, X'$ be two minimal surfaces with $\kappa(X) \geq 0$ and $\kappa(X') \geq 0$, and let $\phi : X \dasharrow X'$ be a birational map. Then $\phi$ is an isomorphism.

**Proof.** Let us show that $\phi$ is a morphism (the proof for $\phi^{-1}$ is the same). Resolve $\phi$ via a sequence of blowups $\pi_i : X_i \to X_{i-1}, X_0 = X$ to obtain a morphism $f : X_n \to X', f = \phi \circ \pi_1 \circ \cdots \circ \pi_n$ with $n$ minimal. If $n = 0$, we are done, so assume $n > 0$. Let $E$ be the exceptional curve of $\pi_n$. If $f(E)$ is a point, then we can factor through $\pi_n$, contradicting minimality. Thus $f(E)$ is a curve $F$. Now, $K_{X'} \cdot F \leq K_{X_n} \cdot E = -1$ where the inequality was proved before for blowups. So there is a curve $F$ with $K_{X'} \cdot F < 0$, implying that $X'$ is ruled and contradicting our hypothesis.

Now, assume that the characteristic of $k$ is neither $2$ nor $3$, and let $X$ have two fibrations $f : X \to B, g : X \to \mathbb{P}^1$ as above. Let $F_b = f^{-1}(b), F_c = g^{-1}(c)$. As before, we show that all the fibers of $g$ are irreducible. The reduced fibers are elliptic curves, and the multiple fibers are multiples of elliptic curves. Let $X = \{ c \in \mathbb{P}^1 \mid F_c \text{ is a multiple fiber of } g \}$. This is a finite set. If $c \in \mathbb{P}^1 \setminus S$, then $f_c = f|_{F_c} : F_c' \to B$ is an étale morphism (using Riemann-Hurwitz, and that the genus of $F_c'$ equals the genus of $B$). $f_c$ induces a homomorphism of algebraic groups $f_c^* : \text{Pic}^0(B) \to \text{Pic}^0(F_c')$ and $\text{Pic}^0(F_c')$ acts canonically on $F_c' \cdot L$ as follows. If $L$ is a degree $0$ line bundle and $x \in F_c'$, then $(L, x) \mapsto y$, where $L \otimes \mathcal{O}_{F_c}(x) \cong \mathcal{O}_{F_c}(y)$. So we get an action of $B$ on $F_c'$ for each $c \in \mathbb{P}^1 \setminus S$. Since $\{f_c\}$ is an algebraic family of homomorphisms of algebraic groups, we get an action $\sigma_0$ of $B$ on $g^{-1}(\mathbb{P}^1 \setminus S) \subset X$. Thus, every element $b \in B$ defines a rational map $X \dasharrow X$, which we can extend to a morphism to get $\sigma : B \times X \to X$. 
