From last time: \( f : X \to B \) is an elliptic/quasi-elliptic fibration, \( F_{b_i} = m_iP_i \) multiple fibers, \( R^1f_*\mathcal{O}_X = L \oplus T \), for \( L \) invertible on \( B \) and \( T \) torsion. \( b \in \text{Supp}(T) \iff h^0(\mathcal{O}_{F_b}) \geq 0 \iff h^1(\mathcal{O}_{F_b}) \geq 2 \iff F_b \) is an exceptional/wild fiber.

**Theorem 1.** With the above notation, \( \omega_X = f^*(L^{-1} \otimes \omega_B) \otimes \mathcal{O}_X(\sum a_iP_i) \), where \( 0 \leq a_i < m, a_i = m_i - 1 \) unless \( F_{b_i} \) is exceptional, and \( \deg(L^{-1} \otimes \omega_B) = 2p_a(B) - 2 + \chi(\mathcal{O}_X) + \ell(T) \), where \( \ell(T) \) is its length as an \( \mathcal{O}_B \)-module.

**Proof.** We have proved most of this: specifically, we have that \( \omega_X = f^*(L^{-1} \otimes \omega_B) \otimes \mathcal{O}_X(\sum a_iP_i) \) for \( 0 \leq a_i < m \). We have a Leray spectral sequence \( E_2^{pq} = H^p(B, R^qf_*\mathcal{O}_X) \Rightarrow H^{p+q}(X, \mathcal{O}_X) \). The smaller order terms give us a short exact sequence

\[
0 \to H^0(\mathcal{O}_B) \to H^1(\mathcal{O}_X) \to H^0(R^1f_*\mathcal{O}_X) \to H^2(\mathcal{O}_B) = 0
\]

\[
0 \to H^2(\mathcal{O}_X) \to H^1(R^1f_*\mathcal{O}_X) \to 0
\]

Using this, we see that

\[
\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X)
\]

\[
= h^0(\mathcal{O}_B) - h^1(\mathcal{O}_B) - h^0(L \oplus T) + h^1(L \oplus T)
\]

\[
= \chi(\mathcal{O}_B) - \chi(L) - h^0(T)
\]

\[
= -\deg L - \ell(T)
\]

by Riemann-Roch, so \( \deg L = -\chi(\mathcal{O}_X) - \ell(T) \). Since \( \deg \omega_B = 2p_a(B) - 2 \), we have \( \deg(L^{-1} \otimes \omega_B) = 2p_a(B) - 2 + \chi(\mathcal{O}_X) + \ell(T) \). It remains to show that \( a_i = m_i - 1 \) if \( F_{b_i} \) is not exceptional. If fact, we can prove something stronger: let \( \alpha_i \) be the order of \( \mathcal{O}_X(P_i) \otimes \mathcal{O}_{P_i} \) in Pic \( (P_i) \). Then we claim that

1. \( \alpha_i \) divides \( m_i \) and \( a_i + 1 \),
2. \( h^0(P_i, \mathcal{O}_{(\alpha_i+1)P_i}) \geq 2 \) and \( h^0(P_i, \mathcal{O}_{\alpha_iP_i}) = 1 \), and
3. \( h^0(P_i, nP_i) \) is a nondecreasing function of \( n \).

Assuming this, if \( a_i < m_i - 1 \), then \( \alpha_i < m_i \), so \( m_iP_i \) is exceptional by (b) and (c), since then \( h^0(\mathcal{O}_{m_iP_i}) \geq 2 \).

We now prove the claim. If \( m > n \geq 1 \), then \( \mathcal{O}_{mP_i} \to \mathcal{O}_{nP_i} \to 0 \) gives \( H^1(P, \mathcal{O}_{mP_i}) \to H^1(P, \mathcal{O}_{nP_i}) \to 0 \), implying that \( n \mapsto h^1(P, \mathcal{O}_{nP_i}) \) is nondecreasing. But by Riemann-Roch and the definition of canonical type, \( \chi(\mathcal{O}_{nP_i}) = 0 \), so
$h^0 = h^1$ is also nondecreasing. Now, by the definition of $\alpha_i$, $\mathcal{O}_X(\alpha_i P_i) \otimes \mathcal{O}_{P_i} \cong \mathcal{O}_{P_i}$, implying that $\mathcal{O}_X(\alpha_i P_i) \otimes \mathcal{O}_{P_i} \cong \mathcal{O}_{P_i}$ as well. We thus obtain an exact sequence $0 \to \mathcal{O}_X(-\alpha_i P_i) \otimes \mathcal{O}_{P_i} = \mathcal{O}_{P_i} \to \mathcal{O}_{(\alpha_i + 1)P_i} \to \mathcal{O}_{\alpha_i P_i} \to 0$, inducing a long exact sequence

$$0 \to k \cong H^0(\mathcal{O}_{P_i}) \to H^0(\mathcal{O}_{(\alpha_i + 1)P_i}) \to H^0(\mathcal{O}_{\alpha_i P_i}) \to \cdots$$

and $h^0(\mathcal{O}_{(\alpha_i + 1)P_i}) \geq 2$. But for $1 \leq j < \alpha_i$, $L_j = \mathcal{O}_X(-j P_i) \otimes \mathcal{O}_{P_i}$ is an invertible $\mathcal{O}_{P_i}$-module whose degree in each component of $P_i$ equals 0. Since $L_j \not\cong \mathcal{O}_{P_i}$, $H^0(L_j) = 0$, and $0 \to L_j \to \mathcal{O}_{(j+1)P_i} \to \mathcal{O}_{j P_i} \to 0$ gives $H^0(\mathcal{O}_{(j+1)P_i}) \cong H^0(\mathcal{O}_{j P_i})$. Since $H^0(\mathcal{O}_{P_i}) \cong k$ for $P$ icoc, $H^0(\mathcal{O}_{2P_i}) \cong \cdots \cong H^0(\mathcal{O}_{\alpha P_i}) \cong k$ as well.

Finally,

$$(\mathcal{O}_X(P_i) \otimes \mathcal{O}_{P_i})^{m_i} \cong \mathcal{O}_X(F_{b_i}) \otimes \mathcal{O}_{P_i} \cong \mathcal{O}_{P_i}$$

This is proved as follows. Since the fiber is cut out by a rational function $f$, $H^0(\mathcal{O}_X(F_{b_i}) \otimes \mathcal{O}_{P_i}) \neq 0$. Via the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(F_{b_i})^{1/f \sim \infty} \to \mathcal{O}_X(F_{b_i}) \otimes \mathcal{O}_{F_{b_i}} \to 0$$

we get a global section of $\mathcal{O}_X(F_{b_i}) \otimes \mathcal{O}_{F_{b_i}}$. But this also has degree 0 along the components. So it must be trivial, but what we proved for icoc. We also have

$$\mathcal{O}_X((\alpha_i + 1)P_i) \otimes \mathcal{O}_{P_i} \cong \omega_X \otimes \mathcal{O}_X(P_i) \otimes \mathcal{O}_{P_i} \cong \omega_{P_i} \cong \mathcal{O}_{P_i}$$

implying that $\alpha_i | a_i + 1$ as desired. $\square$

**Corollary 1.** $K^2 = 0$.

**Corollary 2.** If $h^1(\mathcal{O}_X) \leq 1$, then either $a_i + 1 = m_i$ or $a_i + \alpha_i + 1 = m_i$.

**Proof.** Exercise. $\square$

**Remark.** Raynaud showed that $m_i/\alpha_i$ is a power of $p = \text{char}(k)$ (or is 1 if $\text{char}(k) = 0$). Therefore, there are no exceptional fibers in characteristic 0.

### 1. Classification (contd.)

If $f : X \to B$ is an elliptic/quasi-elliptic fibration, then

$$\omega_X = f^*(L^{-1} \otimes \omega_B) \otimes \mathcal{O}_X(\sum a_i P_i), 0 \leq a_i < m_i$$

If $n \geq 1$ is a multiple of $m_1, \ldots, m_r$, then

$$H^0(X, \omega_X^\otimes n) = H^0(B, L^{-1} \otimes \omega_B^\otimes n \otimes \mathcal{O}_B(\sum a_i(n/m_i)b_i))$$

Now we recall the 4 classes of surfaces:

(a) $\exists$ an integral curve $C$ on $X$ s.t. $K \cdot C < 0$. 
Lemma 1. If $X$ is in (a), then $\kappa(X) = -\infty$, i.e., $p_n = 0$ for all $n \geq 1$. If $X$ is in (b), then $\kappa(X) \leq 0$. If $X$ has an elliptic or quasielliptic fibration $f : X \to B$, and if we let $\lambda(f) = 2p_n(B) - 2 + \chi(\mathcal{O}_X) + \ell(T) + \sum \frac{n_i}{m_i}$, then $X$ is not in class (d) and

- $X$ is in (a) iff $\lambda(f) < 0$, in which case $\kappa(X) = -\infty$,
- $X$ is in (b) iff $\lambda(f) = 0$, in which case $\kappa(X) = 0$,
- $X$ is in (c) iff $\lambda(f) > 0$, in which case $\kappa(X) = 1$.

Proof. If $K \cdot C < 0$, then $X$ is ruled, and $\kappa(X) = \infty$. We did this before, and there is an easy way to see that $p_n = 0$ for all $n \geq 1$. For every divisor $D \in \text{Div}(X), \exists n_D$ s.t. $|D + nK| = \emptyset$ for $n > n_D$. (Since $(D + nK) \cdot C = D \cdot C + n(K \cdot C)$ becomes negative eventually. Now $C$ is effective. We claim that $C^2 \geq 0$, so by our useful lemma, $6|D + nK|$ can’t have an effective divisor. If $C^2 < 0$, then $C \cdot K < 0$ would imply that $C$ was an exceptional curve of the first kind, contradicting the minimality of $X$. Thus, $C^2 \geq 0$.) In particular, $D = K$ gives $|nK| = \emptyset$ for large enough $n$, implying that $|nK| = \emptyset$ for all $n$ (since $p_n < p_m)$.

Next, assume $K \equiv 0$ (case (b)). If $p_n \geq 2$, then dim $|nK| \geq 1 \implies \exists$ a strictly positive divisor $\Delta > 0$ in $|nK|$. Then $\Delta \cdot H > 0$ for a hypersurface section, contradicting $nK \cdot H = 0$ since $K \equiv 0$. So $p_n \leq 1$ for all $n$, implying that $\kappa(X) \leq 0$.

Now assume $X$ has an elliptic/quasielliptic fibration, and let $M = f_*(\omega_X) = L^{-1} \otimes \omega_B$ from last time. Then $M$ has degree $\lambda(f)$. Let $H$ be a very ample divisor on $X$. Then $\pi = f|_H : H \to B$ is some finite map of degree $H \cdot F > 0$. Now $n(K \cdot H) = \deg(\omega_X|_H) = \deg_H(\pi^*M) = (\deg \pi)(\deg_B M) = (H \cdot F)\lambda_f$. So if $\lambda_f < 0$, then $K \cdot H < 0$ and $X$ is in (a).

Similarly, $\lambda(f) = 0 \implies K \cdot H = 0$ for every irreducible hyperplane section $H$, and any curve $C$ can be written, up to $\sim$, as the difference of 2 such. This implies that $K \cdot C = 0 \forall C \implies K \equiv 0$. Lastly, $\lambda(f) > 0 \implies K \cdot C > 0$ for all horizontal irreducible $C$. For vertical $C$, $K \cdot C = 0$ by the formula for $K$, implying that $K \cdot C \geq 0$ for all $C$ integral, $(K^2) = 0$ by the formula, implying that we are in class (c).

Let $X$ be a minimal surface with $K^2 = 0, p_g \leq 1$ (in particular, every surface in class (b) is of this form. Then Noether’s formula gives $10 - 8q + 12p_g = b_2 + 2\Delta$. Since $p_g \leq 1, 0 \leq \Delta \leq 2p_g \leq 2$, also $\Delta = 2(q - s)$ is even, we obtain the following possibilities.

1. $b_2 = 22, b_1 = 0, \chi(\mathcal{O}_X) = 2, q = 0, p_g = 1, \Delta = 0$.
2. $b_2 = 14, b_1 = 2, \chi(\mathcal{O}_X) = 1, q = 1, p_g = 1, \Delta = 0$. 


(3) \( b_2 = 10, b_1 = 0, \chi(\mathcal{O}_X) = 1 \), and either \( q = 0, p_g = 0, \Delta = 0 \) or \( q = 1, p_g = 1, \Delta = 2 \).

(4) \( b_2 = 6, b_1 = 4, \chi(\mathcal{O}_X) = 0, q = 2, p_g = 1, \Delta = 0 \).

(5) \( b_2 = 2, b_1 = 2, \chi(\mathcal{O}_X) = 0 \), and either \( q = 1, p_g = 1, \Delta = 0 \) or \( q = 2, p_g = 0, \Delta = 2 \).

Note. If \( X \) is in class (b) and \( p_g = 1 \), then \( K \sim 0 \) (because \( K = 0, H^0(K) \neq 0 \) imply that \( K \sim 0 \)).

Let’s deal with case 4 of class (b).

**Proposition 1.** Let \( X \) be minimal in class (b), and \( b_2 = 2, b_1 = 2 \). Then \( s = 1, \text{Alb}(X) \text{ is an elliptic curve, and } X \to \text{Alb}(X) \text{ gives an elliptic/quasielliptic fibration.} \)

**Proof.** Let’s see that the fibers of \( f \) are irreducible. If not, we would have \( \rho > 2(F, H, \text{component of } F) \) and \( b_2 \geq \rho > 2 \), contradicting \( b_2 = 2 \). Now, to see that the fibers are not multiple, note that \( \chi(\mathcal{O}_X) = 0 \) from the list.

(9) \[ \deg(L^{-1} \otimes \omega_B) = 2p_a(B) - 2 + \chi(\mathcal{O}_X) + \ell(T) = \ell(T) \geq 0 \]

Since \( \omega_X = f^*(L^{-1} \otimes \omega_B) \otimes \mathcal{O}_X(\sum a_i P_i) \equiv 0 \), we see that \( \ell(T) \cdot f^{-1}(y) + \sum a_i P_i \equiv 0 \). But it is an effective divisor, implying that all the \( a_i = 0, \ell(T) = 0 \) and thus \( a_i = m_i - 1 \forall i \) (there are no wild fibers since \( T = 0 \)). So \( m_i = 1 \forall i \). Thus, we have integral fibers, which is the case of a bielliptic surface. \( \square \)