1. Castelnuovo’s Criterion for Rationality

**Theorem 1.** Any surface with \( q = h^1(X, \mathcal{O}_X) = 0 \) and \( p_2 = h^0(X, \omega_X^{\otimes 2}) = 0 \) is rational.

*Note.* Every rational surface satisfies these: they are birational invariants which vanish for \( \mathbb{P}^2 \).

Reduction 1: Let \( X \) be a minimal surface with \( q = p_2 = 0 \). It is enough to show there is a smooth rational curve \( C \) on \( X \) with \( C^2 \geq 0 \).

*Proof.* First, observe that \( 2g(C) - 2 = C \cdot (C + K) \) and \( \chi(\mathcal{O}_X(C)) = \chi(\mathcal{O}_X) + \frac{1}{2}C(C - K) \). Since \( p_2 = 0 \), \( p_1 = h^0(X, \omega) = h^2(X, \mathcal{O}_X) = 0 \) and \( \chi(\mathcal{O}_X) = 1 \). Since \( h^2(C) = h^0(K - C) \leq h^0(K) = 0 \), \( h^0(C) \geq 1 + \frac{1}{2}C(C - K) \), so \( h^0(C) \geq 2 + C^2 \geq 2 \). Choose a pencil inside this system containing \( C \), i.e. a subspace of dimension 2. The pencil has no fixed component (the only possibility is \( C \), but \( C \) moves in the pencil): after blowing up finitely many base points, we get a morphism \( \tilde{X} \to \mathbb{P}^1 \) with a fiber isomorphic to \( C \cong \mathbb{P}^1 \). Therefore, by the Noether-Enriques theorem, \( \tilde{X} \) is ruled over \( \mathbb{P}^1 \) and \( \tilde{X} \) is rational (as is \( X \)). \( \square \)

Reduction 2: Let \( X \) be a minimal surface with \( q = p_2 = 0 \). It is enough to show that \( \exists \) an effective divisor \( D \) on \( X \) s.t. \( |K + D| = \emptyset \) and \( K \cdot D < 0 \).

*Proof.* This implies that some irreducible component \( C \) of \( D \) satisfies \( K \cdot C < 0 \). Clearly, \( |K + C| \subset |K + D| \). Using Riemann-Roch for \( K + C \) gives

\[
0 = h^0(U + C) + h^0(-C) = h^0(K + C) + h^2(K + C)
\]

\[
\geq 1 + \frac{1}{2}(K + C) \cdot C = g(C)
\]

We thus obtain a smooth, rational curve \( C \) on \( X \): \(-2 = 2g - 2 = C(C + K) \) and \( C \cdot K < 0 \iff C^2 \geq -1 \). Since \( X \) is minimal, \( C^2 \neq -1 \), so \( C^2 \geq 0 \) as desired. \( \square \)

We now prove our second statement. There are three cases:
Case 1 \((K^2 = 0)\): Riemann-Roch gives
\[
h^0(-K) = h^0(-k) + h^0(2K) = h^0(-K) + h^2(-K)
\]
(2)
\[
\geq 1 + \frac{1}{2}K \cdot 2K = 1 + K^2 = 1
\]
so \(|-K| \neq \emptyset\). Take a hyperplane section \(H\) of \(X\). Then there is an \(n \geq 0\) s.t. \(|H + nK| \neq \emptyset\) but \(|H + (n+1)K| = \emptyset\). Since \(-K\) is an effective nonzero divisor, \(H \cdot K < 0\) and \(H \cdot (H + nK)\) is eventually negative and \(H + nK\) is not effective. Let \(D \in |H + nK|\): then \(|D + K| = \emptyset\) and \(K \cdot D = K(H + nK) = K \cdot H < 0\) since \(-K\) is effective, \(H\) very ample.

Case 2 \((K^2 < 0)\): it is enough to find an effective divisor \(E\) on \(X\) s.t. \(K \cdot E < 0\). Then some component \(C\) of \(E\) will have \(K \cdot C < 0\). The genus formula gives
\[-2 \leq 2g - 2 = C(C + K) \implies C^2 \geq -1.\]
\(C^2 = -1\) is impossible since \(X\) is minimal, so \(C^2 \geq 0\). Now \((C + nK) \cdot C\) is negative for \(n \gg 0\), so \(C + nK\) is not effective for \(n \gg 0\) by the useful lemma. So \(\exists n\) s.t. \(|C + nK| \neq \emptyset\) but \(|C + (n+1)K| = \emptyset\). Choosing \(D \in |C + nK|\) gives the desired divisor.

We now find the claimed \(E\). Again, let \(H\) be a hyperplane section: if \(K \cdot H < 0\), we can take \(E = H\); if \(K \cdot H = 0\), we can take \(K + nH\) for \(n \gg 0\); so assume \(K \cdot H > 0\). Let \(\gamma = \frac{-K^2 H}{K^2} > 0\) so that \((H + \gamma K) \cdot K = 0\). Also,
\[
(H + \gamma K)^2 > H^2 + 2\gamma(H \cdot K) + \gamma^2 K = H^2 + \frac{(K \cdot H)^2}{(-K^2)} > 0
\]
So take \(\beta\) rational and slightly larger than \(\gamma\) to get
\[
(H + \beta K) \cdot K < (H + \gamma K) \cdot K = 0
\]
(4)

(since \(K^2 < 0\)) and \((H + \beta K)^2 > 0\). Therefore, \((H + \beta K) \cdot H > 0\). Write \(\beta = \frac{\xi}{r}\). Then
\[
(rH + sK)^2 > 0, (rH + sK) \cdot K < 0, (rH + sK) \cdot H > 0
\]
by equivalent facts for \(\beta\). Let \(D = rH + sK\). For \(m \gg 0\), by Riemann-Roch we get \(h^0(mD) + h^0(K - mD) \geq \frac{1}{2}mD(mD - K) + 1 \to \infty\). Moreover, \(K - mD\) is not effect over for \(m \gg 0\) since \((K - mD) \cdot H = (K \cdot H) - m(D \cdot H)\). Thus, \(mD\) is effective for large \(m\), and we can take \(E \in |mD|\).

Case 3 \((K^2 > 0)\): Assume that there is no such \(D\) as in reduction 2, i.e. \(K \cdot D \geq 0\) for every effective divisor \(D\) s.t. \(|K + D| = \emptyset\). We will obtain a contradiction.

**Lemma 1.** If \(X\) is a minimal surface with \(p_2 = q = 0, K^2 > 0\) and \(K \cdot D \geq 0\) for every effective divisor \(D\) on \(X\) s.t. \(|K + D| = \emptyset\), then

1. \(\text{Pic}(X)\) is generated by \(\omega_X = \mathcal{O}_X(K)\), and the anticanonical bundle \(\mathcal{O}_X(-K)\) is ample. In particular, \(X\) doesn’t have any nonsingular rational curves.
(2) Every divisor of $|-K|$ is an integral curve of arithmetic genus 1.
(3) $(K^2) \leq 5, b_2 \geq 5$. (Here, $b_2 = h^2_{et}(X, \mathbb{Q}_\ell)$ in general.

Proof. First, let us see that every element $D$ of $|-K|$ is an irreducible curve. If not, let $C$ be a component of $D$ s.t. $K \cdot C < 0$ (which we can find, since $K \cdot D = -K^2 < 0$). If $D = C + C', |K + C| = |-D + C| = |-C'| = \emptyset$ since $C'$ is effective. Also, $C \cdot K < 0$, contradicting the hypothesis. So $D$ is irreducible, and similarly $D$ is not a multiple. Furthermore, $p_a(D) = \frac{1}{2}D(D + K) + 1 = 1$, showing (2).

Next, we claim that the only effective divisor s.t. $|D + K| = \emptyset$ is the zero divisor. Assume not, i.e. $\exists D > 0$ s.t. $|K + D| = \emptyset$. Let $x \in D$: then since $h^0(-K) \geq 1 + K^2 \geq 2$, there is a $C \in |-K|$ passing through $x$. $C$ is an integral curve, and cannot be a component of $D$ since then

$$|K + D| \supset |K + C| = |0| \neq \emptyset$$

So $C \cdot D > 0$ since they meet at least in $x$. Then $K \cdot D = -C \cdot D < 0$, contradicting the hypothesis.

As an aside, we claim that $p_n = 0$ for all $n \geq 1$: we know that $p_2 = 0 \implies p_1 = 0$; if $3K$ were effective then $2K$ would be too since $-K$ is effective, which contradicts $p_2 = 0 \implies p_3 = 0$ and by induction $p_n = 0$ for all $n \geq 1$.

We claim that adjunction terminates: if $D$ is any divisor on $X$, then there is an integer $n_D$ s.t. $|D + nK| = \emptyset$ for $n \geq n_D$. To see this, note that $(D + nK) \cdot (-K)$ will eventually become negative. $-K$ is represented by an irreducible curve of positive self-intersection, so by the useful lemma $D + nK$ is not effective for $n >> 0$. Now, let $\Delta$ be an arbitrary effective divisor. Then $\exists n \geq 0$ s.t. $|\Delta + nK| \neq 0$ but $|\Delta + (n + 1)K| = \emptyset$. Take $D \in |\Delta + nK|$ effective. $|D + K| = \emptyset \implies D = 0$ from above. Since any divisor is a difference of effective divisors, Pic $(X)$ is generated by $K$. If $H$ is a hyperplane section on $X$, then $H \sim -nK$ with $k > 0$, implying that $-K$ is ample. Let $C$ be any integral curve on $X$: then $C \sim -mK$ for some $m \geq 1$. $p_a(C) = \frac{1}{2}(-mK)(-mK + K) + 1 = \frac{1}{2}m(m - 1)K^2 + 1 \geq 1$ so there is no smooth rational curve on $X$, completing (1).

We are left to prove (3). Assume that $(K^2) \geq 6$. Then $h^0(-K) \geq 1 + K^2 \geq 7$. Fix points $x$ and $y$ on $X$: we claim that $\exists C \in |-K|$ with $x$ and $y$ singular points of $C$. This would be a contradiction, since $p_a(C) = 1 \implies p_a(\tilde{C}) < 0$ which is absurd. So $K^2 \leq 5$. To see the existence of this $C$, let

$$I_x = \text{Ker } (\mathcal{O}_X \to \mathcal{O}_{X,x}/m_x^2), I_y = \text{Ker } (\mathcal{O}_X \to \mathcal{O}_{X,y}/m_y^2)$$

Then we get, by the Chinese Remainder theorem,

$$0 \to \mathcal{O}_X(-K) \otimes I_x \otimes I_y \to \mathcal{O}_X(-K) \to k^6 \to 0$$

since $\mathcal{O}_{X,x}/m_x^2, \mathcal{O}_{X,y}/m_y^2$ have dimension 3 over $k$. Taking the long exact sequence, we find that $h^0(\mathcal{O}_X(-K) \otimes I_x \otimes I_y) \neq 0$, and get a nonzero section of that sheaf.
It is a divisor of zero passing through \( x \) and \( y \) with multiplicity at least 2, giving us the claimed curve.

Finally, by Noether’s formula, \( 1 = \chi(\mathcal{O}_X) = \frac{1}{12}(K^2 + e(X)) \), where \( e(X) = 2 - 2b_1 + b_2 \). \( b_1 = 2q \) by Hodge theory over \( \mathbb{C} \) (in general, \( B_1 \leq 2q \), but \( q = 0 \implies b_1 = 0 \) as well), so \( 10 = K^2 + b_2 \implies b_2 \geq 5 \). \( \Box \)

We now show that no surface has these properties. In characteristic 0, the Lefschetz principle allows us to reduce to \( k = \mathbb{C} \). Taking the cohomology of the exponential exact sequence \( 0 \to \mathbb{Z} \to \mathcal{O}_X^\text{an} \to (\mathcal{O}_X^\text{an})^* \to 1 \) gives

\[
H^1(\mathcal{O}_X^\text{an}) \to H^1((\mathcal{O}_X^\text{an})^*) \to H^2(X, \mathbb{Z}) \to H^2(\mathcal{O}_X^\text{an}) \to \cdots
\]

By Serre’s GAGA, \( H^i(X, \mathcal{F}) \cong H^i(X^\text{an}, \mathcal{F}^\text{an}) \) for an \( \mathcal{O}_X \)-module \( \mathcal{F} \). Since \( q = p_g = 0, h^1(\mathcal{O}_X^\text{an}) = h^2(\mathcal{O}_X^\text{an}) = 0 \), and

\[
H^1((\mathcal{O}_X^\text{an})^*) \cong H^1(\mathcal{O}_X^*) = \text{Pic} X \cong H^2(X, \mathbb{Z})
\]

This implies that \( b_2 = \text{rank } H^2(X, \mathbb{Z}) = \text{rank } \text{Pic} X = 1 \) contradicting \( b_2 \geq 5 \). For positive characteristic, we will sketch a proof: the first proof was given by Zariski, and the second using étale cohomology by Artin and by Kurke. Our proof will be by reduction to characteristic 0.