Compactly generated spaces

Definition. A space $X$ is said to be **compactly generated** if it satisfies the following condition: A set $A$ is open in $X$ if $A \cap C$ is open in $C$ for each compact subspace $C$ of $X$.

Said differently, a space is compactly generated if its topology is coherent with the collection of compact subspaces of $X$. (See Notes G for a discussion of coherent topologies.) Many spaces are compactly generated; for instance, locally compact spaces are compactly generated, and so are first-countable spaces. (See Lemma 46.3.)

Compactly generated spaces are useful when studying various topologies on the space $C(X,Y)$ of continuous functions $f : X \to Y$, but they occur in other contexts as well. Here we explore their relation to proper maps and to perfect maps.

Definition. A map $f : X \to Y$ is said to be **proper** if for every compact subspace $C$ of $Y$, the subspace $f^{-1}(C)$ of $X$ is compact.

Roughly speaking, $f$ is proper if it does not collapse any subset of $X$ that runs off to infinity onto a compact subspace of $Y$, which does not run off to infinity.

**Theorem K.1.** Let $f : X \to Y$ be a continuous map. If $Y$ is a compactly generated Hausdorff space, and if $f$ is proper, then $f$ is a closed map.

**Proof.** Let $A$ be a closed set in $X$. To show $f(A)$ is closed, we need only to show that $f(A) \cap C$ is closed in $C$ for each compact subspace $C$ of $Y$. Now

$$f(A) \cap C = f(f^{-1}(C) \cap A).$$

The space $f^{-1}(C)$ is compact because $f$ is proper, so its closed subspace $f^{-1}(C) \cap A$ is also compact. The image of this set under $f$ is compact, and is therefore closed in $Y$.

$\square$
Corollary K.2. Let \( f : X \to Y \) be continuous and injective. If \( f \) is proper, and \( Y \) is compactly generated Hausdorff, then \( f \) is an imbedding, whose image is a closed subspace of \( Y \).

Example 1. Let \( f : [0,2\pi] \to \mathbb{R}^2 \) be given by the equation \( f(t) = (\cos t, \sin t) \). Then \( f \) is continuous and injective, and its image is the unit circle, which is closed in \( \mathbb{R}^2 \). However, \( f \) is not proper; the inverse image of the unit circle is not compact. And \( f \) is not an imbedding.

Definition. A map \( f : X \to Y \) is said to be perfect if it is continuous, closed, and surjective, and if \( f^{-1}(y) \) is compact for each \( y \in Y \).

Said differently, a perfect map is a closed quotient map such that the inverse image of each point is compact. Perfect maps have many special properties. For instance, if \( f : X \to Y \) is perfect and \( Y \) is compact, then \( X \) is compact; the same result holds if "compact" is replaced by "paracompact." On the other hand, many "niceness" properties of \( X \) (such as the Hausdorff condition, regularity, local compactness, and second-countability, as well as the condition of being paracompact Hausdorff) are preserved by perfect maps. (See Exercise 12 of §26, Exercise 7 of §31, and Exercise 8 of §41.)

The relation between perfect maps and proper maps is given in the following theorem:

Theorem K.3. Every perfect map is proper. Conversely, let \( f : X \to Y \) be continuous and surjective. If \( f \) is proper, and if \( Y \) is compactly generated Hausdorff, then \( f \) is perfect.

Proof. Suppose \( f \) is a perfect map. Let \( C \) be a compact subspace of \( Y \); let \( \mathcal{U} \) be an open cover of \( f^{-1}(C) \). Given \( y \in C \), the set \( f^{-1}(y) \) can be covered by finitely many elements of \( \mathcal{U} \). Because \( f \) is a closed map, there is a neighborhood \( W \) of \( y \) such that \( f^{-1}(W) \) is covered by these same elements of \( \mathcal{U} \). (See Exercise 6 of §31.) We can cover \( C \) by finitely many such neighborhoods \( W \); then their inverse images cover \( f^{-1}(C) \).
Now we suppose that $f$ is continuous, surjective, and proper, and that $Y$ is compactly generated Hausdorff. The fact that $f^{-1}(\{y\})$ is compact is immediate, since $\{y\}$ is compact. The fact that $f$ is closed follows from Theorem K.1. \[\]

It is an interesting fact that if $X$ is not compactly generated, it may be given a (finer) topology that is compactly generated, and has exactly the same collection of compact subspaces:

Theorem K.4. Let $X_T$ be a space with underlying set $X$ and topology $T$. There is a unique topology $C$ on $X$, finer than $T$, such that:

(i) If $D$ is a subset of $X$, and if $D$ is compact in the topology it inherits from $X_T$, or if $D$ is compact in the topology it inherits from $X_C$, then these two topologies on $D$ are the same.

(ii) $X_C$ is compactly generated.

Proof. Let $\{C_\alpha\}$ be the family of compact subspaces of $X_T$. In view of Theorem K.4, there is a topology $C$ on $X$, finer than $T$, such that each space $C_\alpha$ is a subspace of $X_C$ and the topology of $X_C$ is coherent with the subspaces $C_\alpha$.

Suppose $D$ is compact in the topology it inherits from $X_T$. Then in this topology, it is one of the spaces $C_\alpha$; and as just noted, each space $C_\alpha$ is a subspace of $X_C$.

Now suppose $D$ is compact in the topology it inherits from $X_C$. Because the identity map $i: X_C \rightarrow X_T$ is continuous, $D$ is compact in the topology it inherits from $X_T$. Then the previous paragraph applies.

The space $X_C$ is compactly generated. For by definition, $U$ is open in $X_C$ if and only if $U \cap D$ is open in $D$ for each compact subspace $D$ of $X_T$. But $D$ is a compact subspace of $X_T$ if and only if it is a compact subspace of $X_C$.

To prove uniqueness, let $C'$ be any topology satisfying the conditions of the theorem. Because $X_C'$ is compactly generated, a set $U$ is open in $X_C'$, if and only if $U \cap D$ is open in $D$ for each compact subspace $D$ of $X_C'$. By (i), $D$ is a compact subspace of $X_C'$ if and only if it is a compact subspace of $X_T$. Therefore a set $U$ is open in $X_C'$ if and only if it is open in $X_C$. \[\]
The class of compactly generated spaces is an interesting one to explore. Like the class of normal spaces, it is not closed under the operations of taking subspaces or products. We shall show that if \( J \) is uncountable, then \( \mathbb{R}^J \) is not compactly generated. It follows that the subspace \((0,1)^J\) of \([0,1]^J\) is not compactly generated, although \([0,1]^J\) is compact and thus compactly generated. It also follows that an arbitrary product of compactly generated spaces need not be compactly generated. (The same is true for finite products, but the required example is more complicated. See [D], p.249.)

**Example 2.** If \( J \) is uncountable, then \( \mathbb{R}^J \) is not compactly generated. (This example is adapted from [Wd].)

Given \( n \geq 1 \), let \( A_n \) be the set of all points \( x \) of \( \mathbb{R}^J \) such that \( x_\alpha = 0 \) for at most \( n \) values of \( \alpha \), and \( x_\alpha = n \) for all other values of \( \alpha \). We show that each set \( A_n \) is closed in \( \mathbb{R}^J \):

Let \( p \) be a point of \( \mathbb{R}^J \) not in \( A_n \). If \( p_\beta \in \{0,n\} \) for some \( \beta \), let \( U \) be a neighborhood of \( p_\beta \) not containing 0 or \( n \); then \( \prod_\beta^{-1}(U) \) is a neighborhood of \( p \) disjoint from \( A_n \). If \( p_\alpha \notin \{0,n\} \) for all \( \alpha \), then since \( p \notin A_n \), there must be a finite set \( J_0 \) of indices containing more than \( n \) elements such that \( p_\alpha = 0 \) for \( \alpha \in J_0 \). Setting \( U_\alpha = (-1,1) \) for \( \alpha \in J_0 \) and \( U_\alpha = \mathbb{R} \) otherwise, we obtain a neighborhood \( \prod U_\alpha \) of \( p \) disjoint from \( A_n \).

We now show that if \( C \) is a compact subspace of \( \mathbb{R}^J \), then \( C \) intersects only finitely many of the sets \( A_n \). Since \( C \) is compact, so is \( \prod A_\alpha \); therefore the latter is contained in some closed interval \([-n_\alpha',n_\alpha']\) of \( \mathbb{R} \). Then \( C \) lies in \( \prod A_\alpha \); we show that \( \prod A_\alpha \) intersects only finitely many sets \( A_n \). Since \( J \) is uncountable, the map \( \alpha \mapsto n_\alpha \) of \( J \) into \( \mathbb{Z}_+ \) must map some infinite subset \( J_0 \) of \( J \) to a single integer \( N \). It follows that \( A_n \) does not intersect \( \prod A_\alpha \) if \( n > N \). For if \( x \) belongs to \( A_n \), then all but finitely many \( x_\alpha \) are greater than \( N \), while if \( x \) belongs to \( \prod A_\alpha \), infinitely many \( |x_\alpha| \) must be less than or equal to \( N \).
Let $T$ be the union of the sets $A_n$. We show that $T$ is not closed in $\mathbb{R}^J$, but that $T \cap C$ is closed in $C$ for every compact subspace $C$ of $\mathbb{R}^J$.

It is easy to see that $T$ is not closed in $\mathbb{R}^J$, for $0$ is a limit point of $T$ that is not in $T$. Given a basis element $\bigcup U_\alpha$ containing $0$, let $J_0$ be the (finite) set of indices $\alpha$ for which $U_\alpha \neq \mathbb{R}$. If we set $x_\alpha = 0$ for $\alpha \in J_0$ and $x_\alpha = n$ otherwise (where $n$ is the number of elements in $J_0$), we obtain a point $x$ of $T$ that lies in $\bigcup U_\alpha$.

It is also easy to see that $T \cap C$ is closed in $C$ if $C$ is a compact subspace of $\mathbb{R}^J$. For in this case $T \cap C$ is the union of finitely many sets of the form $A_n \cap C$; and since $A_n$ is closed in $\mathbb{R}^J$, the set $A_n \cap C$ is closed in $C$. 