10 Excision and applications

We have found two general properties of singular homology: homotopy invariance and the long exact sequence of a pair. We also claimed that $H_\ast(X, A)$ “depends only on $X - A$.” You have to be careful about this. The following definition gives conditions that will capture the sense in which the relative homology of a pair $(X, A)$ depends only on the complement of $A$ in $X$.

**Definition 10.1.** A triple $(X, A, U)$ where $U \subseteq A \subseteq X$, is excisive if $\overline{U} \subseteq \text{Int}(A)$. The inclusion $(X - U, A - U) \subseteq (X, A)$ is then called an excision.

**Theorem 10.2.** An excision induces an isomorphism in homology,

$$H_\ast(X - U, A - U) \cong H_\ast(X, A).$$
So you can cut out closed bits of the interior of $A$ without changing the relative homology. The proof will take us a couple of days. Before we give applications, let me pose a different way to interpret the motto “$H_\ast(X, A)$ depends only on $X - A$.” Collapsing the subspace $A$ to a point gives us a map of pairs

$$(X, A) \to (X/A, \ast).$$

When does this map induce an isomorphism in homology? Excision has the following consequence.

**Corollary 10.3.** Assume that there is a subspace $B$ of $X$ such that (1) $\overline{A} \subseteq \text{Int}B$ and (2) $A \to B$ is a deformation retract. Then

$$H_\ast(X, A) \to H_\ast(X/A, \ast)$$

is an isomorphism.

**Proof.** The diagram of pairs

$$
\begin{array}{ccc}
(X, A) & \xrightarrow{i} & (X, B) \\
\downarrow & & \downarrow \\
(X/A, \ast) & \xrightarrow{i} & (X/A, B/A) \\
\downarrow & & \downarrow \\
(X/A, \ast) & \xleftarrow{j} & (X/A - \ast, B/A - \ast)
\end{array}
$$

commutes. We want the left vertical to be a homology isomorphism, and will show that the rest of the perimeter consists of homology isomorphisms. The map $i$ is a homeomorphism of pairs while $j$ is an excision by assumption (1). The map $i$ induces an isomorphism in homology by assumption (2), the long exact sequences, and the five-lemma. Since $I$ is a compact Hausdorff space, the map $B \times I \to B/A \times I$ is again a quotient map, so the deformation $B \times I \to B$, which restricts to the constant deformation on $A$, descends to show that $\ast \to B/A$ is a deformation retract. So the map $\tau$ is also a homology isomorphism. Finally, $\overline{A} \subseteq \text{Int}(B/A)$ in $X/A$, by definition of the quotient topology, so $\tau$ induces an isomorphism by excision.

Now what are some consequences? For a start, we’ll finally get around to computing the homology of the sphere. It happens simultaneously with a computation of $H_\ast(D^n, S^{n-1})$. (Note that $S^{-1} = \emptyset$.) To describe generators, for each $n \geq 0$ pick a homeomorphism

$$(\Delta^n, \partial\Delta^n) \to (D^n, S^{n-1}) ,$$

and write

$$\iota_n \in S_n(D^n, S^{n-1})$$

for the corresponding relative $n$-chain.

**Proposition 10.4.** Let $n > 0$ and let $\ast \in S^{n-1}$ be any point. Then:

$$H_q(S^n) = \begin{cases} \mathbb{Z} = \langle [\iota_{n+1}] \rangle & \text{if } q = n > 0 \\ \mathbb{Z} = \langle [\iota_0] \rangle & \text{if } q = 0, n > 0 \\ \mathbb{Z} \oplus \mathbb{Z} = \langle [\iota_0], [\partial\iota] \rangle & \text{if } q = n = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_q(D^n, S^{n-1}) = \begin{cases} \mathbb{Z} = \langle [\iota_n] \rangle & \text{if } q = n \\ 0 & \text{otherwise} \end{cases}.$$
Proof. The division into cases for $H_q(S^n)$ can be eased by employing reduced homology. Then the claim is merely that for $n \geq 0$

$$\tilde{H}_q(S^{n-1}) = \begin{cases} \mathbb{Z} & \text{if } q = n - 1 \\ 0 & \text{if } q \neq n - 1 \end{cases}$$

and the map

$$\partial : H_q(D^n, S^{n-1}) \to \tilde{H}_{q-1}(S^{n-1})$$

is an isomorphism. The second statement follows from the long exact sequence in reduced homology together with the fact that $\tilde{H}_*(D^n) = 0$ since $D^n$ is contractible. The first uses induction and the pair of isomorphisms

$$\tilde{H}_{q-1}(S^{n-1}) \cong H_q(D^n, S^{n-1}) \cong H_q(D^n/S^{n-1}, \ast)$$

since $D^n/S^{n-1} \cong S^n$. The right hand arrow is an isomorphism since $S^{n-1}$ is a deformation retract of a neighborhood in $D^n$.

Why should you care about this complicated homology calculation?

**Corollary 10.5.** If $m \neq n$, then $S^m$ and $S^n$ are not homotopy equivalent.

*Proof.* Their homology groups are not isomorphic.

**Corollary 10.6.** If $m \neq n$, then $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.

*Proof.* If $m$ or $n$ is zero, this is clear, so let $m, n > 0$. Assume we have a homeomorphism $f : \mathbb{R}^m \to \mathbb{R}^n$. This restricts to a homeomorphism $\mathbb{R}^m - \{0\} \to \mathbb{R}^n - \{f(0)\}$. But these spaces are homotopy equivalent to spheres of different dimension.

**Theorem 10.7** (Brouwer fixed-point theorem). If $f : D^n \to D^n$ is continuous, then there is some point $x \in D^n$ such that $f(x) = x$.

*Proof.* Suppose not. Then you can draw a ray from $f(x)$ through $x$. It meets the boundary of $D^n$ at a point $g(x) \in S^{n-1}$. Check that $g : D^n \to S^{n-1}$ is continuous. If $x$ is on the boundary, then $x = g(x)$, so $g$ provides a factorization of the identity map on $S^{n-1}$ through $D^n$. This is inconsistent with our computation because the identity map induces the identity map on $\tilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$, while $\tilde{H}_{n-1}(D^n) = 0$.
Our computation of the homology of a sphere also implies that there are many non-homotopic self-maps of $S^n$, for any $n \geq 1$. We will distinguish them by means of the “degree”: A map $f : S^n \to S^n$ induces an endomorphism of the infinite cyclic group $H_n(S^n)$. Any endomorphism of an infinite cyclic group is given by multiplication by an integer. This integer is well defined (independent of a choice of basis), and any integer occurs. Thus $\text{End}(\mathbb{Z}) = \mathbb{Z}_\times$, the monoid of integers under multiplication. The homotopy classes of self-maps of $S^n$ also form a monoid, under composition, and:

**Theorem 10.8.** Let $n \geq 1$. The degree map provides us with a surjective monoid homomorphism

$$\deg : [S^n, S^n] \to \mathbb{Z}_\times.$$ 

**Proof.** Degree is multiplicative by functoriality of homology.

We construct a map of degree $k$ on $S^n$ by induction on $n$. If $n = 1$, this is just the winding number; an example is given by regarding $S^1$ as unit complex numbers and sending $z$ to $z^k$. The proof that this has degree $k$ is an exercise.

Suppose we’ve constructed a map $f_k : S^{n-1} \to S^{n-1}$ of degree $k$. Extend it to a map $\overline{f}_k : D^n \to D^n$ by defining $\overline{f}_k(tx) = tf(x)$ for $t \in [0, 1]$. We may then collapse the sphere to a point and identify the quotient with $S^n$. This gives us a new map $g_k : S^n \to S^n$ making the diagram below commute.

$$
\begin{array}{ccc}
H_{n-1}(S^{n-1}) & \cong & H_n(D^n, S^{n-1}) \\
\downarrow f_\ast & & \downarrow g_\ast \\
H_{n-1}(S^{n-1}) & \cong & H_n(D^n, S^n)
\end{array}
$$

The horizontal maps are isomorphisms, so $\deg g_k = k$ as well. 

We will see (in 18.906) that this map is in fact an isomorphism.
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