19. COEFFICIENTS

Abelian groups can be quite complicated, even finitely generated ones. Vector spaces over a field are so much simpler! A vector space is determined up to isomorphism by a single cardinality, its dimension. Wouldn’t it be great to have a version of homology that took values in the category of vector spaces over a field?

We can do this, and more. Let \( R \) be any commutative ring at all. Instead of forming the free abelian group on \( \mathrm{Sin}_*(X) \), we could just as well form the free \( R \)-module:

\[
S_*(X; R) = R\mathrm{Sin}_*(X)
\]

This gives, first, a simplicial object in the category of \( R \)-modules. Forming the alternating sum of the face maps produces a chain complex of \( R \)-modules: \( S_n(X; R) \) is an \( R \)-module for each \( n \), and \( d : S_n(X; R) \to S_{n-1}(X; R) \) is an \( R \)-module homomorphism. The homology groups are then again \( R \)-modules:

\[
H_n(X; R) = \frac{\ker(d : S_n(X; R) \to S_{n-1}(X; R))}{\text{im}(d : S_{n+1}(X; R) \to S_n(X; R))}.
\]

This is the singular homology of \( X \) with coefficients in the commutative ring \( R \). It satisfies all the Eilenberg-Steenrod axioms, with

\[
H_n(\ast; R) = \begin{cases} R & \text{for } n = 0 \\ 0 & \text{otherwise.} \end{cases}
\]

(We could actually have replaced the ring \( R \) by any abelian group here, but this will become much clearer after we have the tensor product as a tool.) This means that all the work we have done for “integral homology” carries over to homology with any coefficients. In particular, if \( X \) is a
CHAPTER 2. COMPUTATIONAL METHODS

CW complex we have the cellular homology with coefficients in \( R, C_s(X;R) \), and its homology is isomorphic to \( H_*(X;R) \).

The coefficient rings that are most important in algebraic topology are simple ones: the integers and the prime fields \( \mathbb{F}_p \) and \( \mathbb{Q} \); almost always a PID.

As an experiment, let’s compute \( H_*(\mathbb{R}P^n;R) \) for various rings \( R \). Let’s start with \( R = \mathbb{F}_2 \), the field with 2 elements. This is a favorite among algebraic topologists, because using it for coefficients eliminates all sign issues. The cellular chain complex has \( C_*(\mathbb{R}P^n;\mathbb{F}_2) = \mathbb{F}_2 \) for \( 0 \leq k \leq n \), and the differential alternates between multiplication by 2 and by 0. But in \( \mathbb{F}_2, 2 = 0 \): so \( d = 0 \), and the cellular chains coincide with the homology:

\[
H_k(\mathbb{R}P^n;\mathbb{F}_2) = \begin{cases} 
\mathbb{F}_2 & \text{for } 0 \leq k \leq n \\
0 & \text{otherwise} 
\end{cases}
\]

On the other hand, suppose that \( R \) is a ring in which 2 is invertible. The universal case is \( \mathbb{Z}[1/2] \), but any subring of the rationals containing \( 1/2 \) would do just as well, as would \( \mathbb{F}_p \) for \( p \) odd. Now the cellular chain complex (in dimensions 0 through \( n \)) looks like

\[
R \xleftarrow{0} R \xleftarrow{2} R \xleftarrow{0} R \xleftarrow{2} \cdots \xleftarrow{2} R
\]

for \( n \) even, and

\[
R \xleftarrow{0} R \xleftarrow{2} R \xleftarrow{0} R \xleftarrow{2} \cdots \xleftarrow{0} R
\]

for \( n \) odd. Therefore for \( n \) even

\[
H_k(\mathbb{R}P^n;R) = \begin{cases} 
R & \text{for } k = 0 \\
0 & \text{otherwise} 
\end{cases}
\]

and for \( n \) odd

\[
H_k(\mathbb{R}P^n;R) = \begin{cases} 
R & \text{for } k = 0 \\
R & \text{for } k = n \\
0 & \text{otherwise} 
\end{cases}
\]

You get a much simpler result: Away from 2, even projective spaces look like points, and odd projective spaces look like spheres!

I’d like to generalize this process a little bit, and allow coefficients not just in a commutative ring, but more generally in a module \( M \) over a commutative ring; in particular, any abelian group. This is most cleanly done using the mechanism of the tensor product. That mechanism will also let us address the following natural question:

**Question 19.1.** Given \( H_*(X;R) \), can we deduce \( H_*(X;M) \) for an \( R \)-module \( M \)?

The answer is called the “universal coefficient theorem”. I’ll spend a few days developing what we need to talk about this.