18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.
The T-functor and Unstable Algebras (Lecture 20)

Our first order of business is to prove the following assertion, which was stated without proof in the previous lecture:

**Lemma 1.** Fix an integer $n$. Then for $p \gg 0$, the tensor product $\Phi^p F(1) \otimes F(n)$ is generated by a single element.

**Proof.** We may identify $F(n)$ with the subspace of $F_2[t_1, \ldots, t_n]$ spanned by those polynomials which are symmetric and additive in each variable. The module $\Phi^p F(1)$ can similarly be identified with the subspace of $F_2[t]$ spanned by those polynomials of the form $\{t^{2^k}\}_{k \geq p}$. We wish to show that the tensor product $\Phi^p F(1) \otimes F(n)$ is generated by the element $t^{2^p} \otimes (t_1 \ldots t_n)$. This element determines a map

$$F(n + 2^p) \to \Phi^p F(1) \otimes F(n);$$

it will therefore suffice to show that $\beta$ is surjective. The right hand side has a basis consisting of expressions of the form

$$t^{2^p+q} \otimes \sigma(t_1^{2^{a_1}} \ldots t_n^{2^{a_n}}),$$

where $\sigma$ denotes the operation of symmetrization. We now observe that this basis element is the image of

$$\sigma(t_1^{2^{a_1}} \ldots t_n^{2^{a_n}} t_{n+1}^{2^{a_{n+1}}} \ldots t_{n+2p}^{2^{a_{n+2p}}}) \in F(n + 2^p)$$

provided that $2^p > n$. \qed

In the last lecture, we saw that Lemma 1 implies that Lannes’ T-functor $T_V$ commutes with tensor products. It follows that $M$ is an unstable $A$-module equipped with a multiplication map $M \otimes M \to M$, then $T_V(M)$ inherits a multiplication

$$T_V M \otimes T_V M \simeq T_V (M \otimes M) \to T_V M.$$

**Proposition 2.** Suppose that $M$ is an unstable $A$-algebra. Then the multiplication defined above endows $T_V M$ with the structure of an unstable $A$-algebra.

**Proof.** Since $M$ is commutative, associative, and unital, we deduce immediately that $T_V M$ has the same properties. The only nontrivial point is to verify that $\text{Sq}^{\deg(x)}(x) = x^2$ for every homogeneous element $x \in T_V M$. Before proving this, we indulge in a slight digression.

Let $M$ be an unstable $A$-module. There is a canonical map $f^*_M : \Phi M \to \text{Sym}^2 M$, given by the formula

$$\Phi(x) \mapsto x^2.$$ 

By definition, an unstable $A$-algebra is an unstable $A$-module $M$ equipped with a commutative, associative, and unital multiplication $m : M \otimes M \to M$ such that the diagram

$$
\begin{array}{ccc}
\Phi M & \xrightarrow{f^*_M} & M \\
\downarrow{f_M} & & \downarrow{m} \\
\text{Sym}^2 M & & \\
\end{array}
$$

commutes.
commutes. Here \( f_M : \Phi M \rightarrow M \) is the map described by the formula \( x \mapsto \text{Sq}^\text{deg}(x)x \).

Applying \( T_V \) to the commutative diagram above, we get a new commutative diagram

\[
\begin{array}{ccc}
T_V \Phi M & \xrightarrow{T_V f_M} & T_V M \\
\downarrow & & \downarrow \\
T_V \text{Sym}^2 M & \xrightarrow{T_V f'_M} & T_V \text{Sym}^2 M
\end{array}
\]

Since the functor \( T_V \) preserves colimits and tensor products, we have a canonical isomorphism \( \alpha : T_V \text{Sym}^2 M \simeq \text{Sym}^2 T_V M \); similarly we have an identification \( \beta : T_V \Phi M \simeq \Phi T_V M \). Under the isomorphism \( \alpha \), the map \( T_V M \) corresponds to the multiplication map \( \text{Sym}^2 T_V M \rightarrow T_V M \) given by the ring structure on \( T_V M \). To prove that \( T_V M \) is an unstable \( A \)-algebra, it will suffice to show that the maps \( T_V f_M \) and \( T_V f'_M \) can be identified, by means of \( \alpha \) and \( \beta \), with \( f_{T_V M} \) and \( f'_{T_V M} \), respectively. We will give a proof for \( f'_{T_V M} \), leaving the first case as an exercise to the reader.

We wish to show that the diagram

\[
\begin{array}{ccc}
T_V \Phi M & \xrightarrow{T_V f'_M} & T_V \text{Sym}^2 M \\
\downarrow & \alpha & \downarrow \\
\Phi T_V M & \xrightarrow{f'_{T_V M}} & \text{Sym}^2 T_V M
\end{array}
\]

is commutative. Using the definition of \( T_V \), we are reduced to proving that the adjoint diagram

\[
\begin{array}{ccc}
\Phi M & \xrightarrow{f'_M} & \text{Sym}^2 M \\
\downarrow & & \downarrow \\
(\Phi T_V M) \otimes \text{H}^*(BV) & \xrightarrow{} & (\text{Sym}^2 T_V M) \otimes \text{H}^*(BV).
\end{array}
\]

To prove this, we consider the larger diagram

\[
\begin{array}{ccc}
\Phi M & \xrightarrow{f'_M} & \text{Sym}^2 M \\
\downarrow & & \downarrow \\
\Phi(T_V M \otimes \text{H}^*(BV)) & \xrightarrow{} & \text{Sym}^2 (T_V M \otimes \text{H}^*(BV)) \\
\downarrow & \sim & \downarrow \\
\Phi(T_V M) \otimes \text{H}^*(BV) & \xrightarrow{} & \text{Sym}^2 (T_V M) \otimes \text{Sym}^2 \text{H}^*(BV) \\
\downarrow & & \downarrow \\
\Phi(T_V M) \otimes \text{H}^*(BV) & \xrightarrow{} & \text{Sym}^2 (T_V M) \otimes \text{H}^*(BV).
\end{array}
\]

The top square obviously commutes. The middle square commutes because the construction of the map \( f'_M \) is compatible with the formation of tensor products in \( M \). The lower square commutes because \( \text{H}^*(BV) \) is an unstable \( A \)-algebra. It follows that the outer square commutes as well, as desired. \( \square \)

Let \( M \) be an unstable \( A \)-algebra, so that \( T_V M \) inherits the structure of an unstable \( A \)-algebra. We now characterize \( T_V M \) by a universal property.
Proposition 3. Let $\mathcal{K}$ denote the category of unstable $A$-algebras. For every pair of objects $M, N \in \mathcal{K}$, the image of the inclusion

$$\text{Hom}_\mathcal{K}(TVM, N) \subseteq \text{Hom}_A(TVM, N) \simeq \text{Hom}_A(M, N \otimes H^*(BV))$$

consists of those maps $M \to N \otimes H^*(BV)$ which are compatible with the ring structure.

Proof. We will show that a map $u : TVM \to N$ is compatible with multiplication if and only if the adjoint map $v : M \to N \otimes H^*(BV)$ is compatible with multiplication; an analogous (but easier) argument shows that $u$ is unital if and only if $v$ is unital.

By definition, $u$ is compatible with multiplication if and only if the diagram

$$\begin{array}{c}
(TVM) \otimes (TVM) \\
\downarrow \downarrow \\
TV(M \otimes M) \\
\downarrow u \\
N \otimes N
\end{array}$$

is commutative. This is equivalent to the commutativity of the adjoint diagram

$$\begin{array}{c}
TV(M \otimes TVM) \otimes H^*(BV) \\
\downarrow w_0 \\
M \otimes M \\
\downarrow v \\
M \otimes N \otimes H^*(BV)
\end{array} \xrightarrow{w_1} \begin{array}{c}
N \otimes N \otimes H^*(BV) \\
\downarrow w_2 \\
N \otimes H^*(BV)
\end{array}$$

To prove that this is equivalent to the assumption that $v$ is compatible with multiplication, it will suffice to show that the composition $w_2 \circ w_1 \circ w_0$ coincides with the composition

$$M \otimes M \xrightarrow{v \otimes u} (N \otimes H^*(BV)) \otimes (N \otimes H^*(BV)) \to N \otimes H^*(BV).$$

This follows from the commutativity of the diagram

$$\begin{array}{c}
(TVM \otimes H^*(BV)) \otimes (TVM \otimes H^*(BV)) \\
\downarrow \downarrow \downarrow \downarrow \\
TVM \otimes TVM \otimes H^*(BV) \\
\downarrow \downarrow \\
N \otimes N \otimes H^*(BV) \\
\downarrow \\
N \otimes H^*(BV)
\end{array} \xrightarrow{u \otimes u} \begin{array}{c}
(N \otimes H^*(BV)) \otimes (N \otimes H^*(BV)) \\
\downarrow \\
N \otimes H^*(BV)
\end{array}$$

Corollary 4. Regarded as a functor from $\mathcal{K}$ to itself, Lannes’ $T$-functor is left adjoint to the functor $N \mapsto N \otimes H^*(BV)$. 
Corollary 5. Let $F_{\text{Alg}}(n)$ denote the free unstable $A$-algebra on one generator in degree $n$. Then we have a canonical isomorphism of unstable $A$-algebras

$$TF_{\text{Alg}}(n) \simeq F_{\text{Alg}}(n) \otimes \ldots \otimes F_{\text{Alg}}(0).$$

Proof. Let $M$ be an arbitrary unstable $A$-algebra. Then

$$\text{Hom}_{\mathcal{K}}(TF_{\text{Alg}}(n), M) \simeq \text{Hom}_{\mathcal{K}}(F_{\text{Alg}}(n), M \otimes F_2[t]) \simeq (M \otimes F_2[t])^n \simeq M^n \times M^{n-1} \times \ldots \times M^0 \simeq \text{Hom}_{\mathcal{K}}(F_{\text{Alg}}(n), M) \times \ldots \times \text{Hom}_{\mathcal{K}}(F_{\text{Alg}}(0), M) \simeq \text{Hom}_{\mathcal{K}}(F_{\text{Alg}}(n) \otimes \ldots \otimes F_{\text{Alg}}(0), M).$$

Recall that $F_{\text{Alg}}(n)$ can be identified with the cohomology of the Eilenberg-MacLane space $K(F_2, n)$. Similarly, the Kunneth theorem allows us to identify the tensor product $F_{\text{Alg}}(n) \otimes \ldots \otimes F_{\text{Alg}}(0)$ with the cohomology of the product

$$K(F_2, n) \times K(F_2, n-1) \times \ldots \times K(F_2, 0) \simeq K(F_2, n)^{BF_2}. $$

The isomorphism of Corollary 5 is induced by the canonical map

$$\eta_X : TV^*X \rightarrow H^*(X^{BV})$$

in the special case where $X = K(F_2, n)$ and $V = F_2$. We may therefore restate Corollary 5 in the following more conceptual form: if $X$ is an Eilenberg-MacLane space $K(F_2, n)$ and $V = F_2$, then the map $\eta_X$ is an isomorphism. Our next goal in this course is to prove this statement for a much larger class of spaces.