In this lecture, we will discuss the relationship between the category $\mathcal{S}_p^\vee$ of $p$-profinite spaces and the usual category $\mathcal{S}$ of spaces. As we have seen earlier, there is a pair of adjoint functors

$$\mathcal{S} \xrightarrow{\lim} \mathcal{S}_p^\vee.$$ 

The composition

$$X \mapsto \lim X^\vee$$

is a functor from the category of spaces to itself. We will denote this functor by $X \mapsto \hat{X}$. We think of this functor as "$p$-adically completing" the homotopy type of $X$. The following assertion makes this idea precise:

**Theorem 1.** Let $X$ be a simply connected space, and assume that every homotopy group $\pi_i X$ is finitely generated (as an abelian group). Then $\hat{X}$ is again simply connected, and the unit map $X \rightarrow \hat{X}$ induces isomorphisms

$$\pi_i X \otimes \mathbb{Z}_p \simeq \pi_i \hat{X},$$

where $\mathbb{Z}_p$ denotes the ring of $p$-adic integers.

We will reduce the proof of Theorem 1 to the following calculation:

**Lemma 2.** For each $i \geq 0$, the canonical map

$$H_i K(\mathbb{Z}, 1) \rightarrow \lim H_i K(\mathbb{Z}/p^k \mathbb{Z}, 1)$$

is an isomorphism in the category of pro-$\mathbb{F}_p$-vector spaces.

**Proof.** If $i \leq 1$, then the pro-system on the right is constant (and isomorphic to the $H_i K(\mathbb{Z}, 1)$). If $i > 1$, then the homology group on the left vanishes, and the inverse system on the right can be identified with the system

$$\ldots \rightarrow \mathbb{F}_p \xrightarrow{0} \mathbb{F}_p \xrightarrow{0} \mathbb{F}_p,$$

which is trivial as a pro-vector space. \hfill \Box

**Corollary 3.** For each $i \geq 0$ and each $n > 0$, the canonical map

$$\phi : H_i K(\mathbb{Z}, n) \rightarrow \lim H_i K(\mathbb{Z}/p^k \mathbb{Z}, n)$$

is an isomorphism in the category of pro-$\mathbb{F}_p$-vector spaces.

**Proof.** We work by induction on $n$, the case $n = 1$ having been handled above. For every abelian group $A$, the Eilenberg-Moore spectral sequence has $E_2$-term given by

$$E_2^{a,b}(A) \simeq \text{Tor}_a^H(K(A, n-1), (\mathbb{F}_p, \mathbb{F}_p)_b).$$
and converges to $H_n K(A, n)$. It follows from the inductive hypothesis that the canonical map
\[
E_2^{a,b}(\mathbb{Z}) \to "\lim E_2^{a,b}(\mathbb{Z}/p^k\mathbb{Z})"
\]
duces an isomorphism of pro-vector spaces for each $a, b$. It follows that we get an isomorphism of pro-vector spaces at the $E_\infty$-term. The convergence of the spectral sequence then implies that $\phi$ is an isomorphism of pro-vector spaces.

**Corollary 4.** For each $i \geq 0$ and each $n > 0$, the canonical map
\[
\lim H^* K(\mathbb{Z}/p^k\mathbb{Z}, n) \to H^* K(\mathbb{Z}, n)
\]
is an isomorphism of $F_p$-vector spaces.

**Corollary 5.** Let $X = K(\mathbb{Z}, n)$, where $n \geq 1$. Then the $p$-profinite completion $X^\wedge$ can be identified with the formal inverse limit
\[
y = "\lim K(\mathbb{Z}/p^k\mathbb{Z}, n)".
\]

**Proof.** We have a canonical map $X^\wedge \to Y$ of $p$-profinite spaces. To show that it is a homotopy equivalence, it will suffice to show that it induces an isomorphism on cohomology. This follows immediately from Corollary 4.

**Corollary 6.** If $X = K(\mathbb{Z}, n)$, then the canonical map $\tilde{X} \to K(\mathbb{Z}, 1)$ is a homotopy equivalence.

The following result will allow us to promote this result to more general Eilenberg-MacLane spaces:

**Lemma 7.** Let $X$ and $Y$ be spaces such that $H^*(X; F_p)$ and $H^*(Y; F_p)$ are finite dimensional in each degree. Then the canonical map $\tilde{X} \times \tilde{Y} \to \tilde{X} \times \tilde{Y}$ is a homotopy equivalence.

**Proof.** Since the functor $\lim : \mathcal{S}_p \to \mathcal{S}$ preserves homotopy limits, it will suffice to show that the canonical map $(X \times Y)^\wedge \to X^\wedge \times Y^\wedge$ is an equivalence of $p$-profinite spaces. For this, it suffices to show that this map induces an isomorphism on cohomology. In general, we have isomorphisms
\[
H^*(X^\wedge \times Y^\wedge) \simeq H^*(X^\wedge) \otimes H^*(Y^\wedge) \simeq H^*(X) \otimes H^*(Y)
\]
If the cohomology groups of $X$ and $Y$ are finite dimensional in each degree, then the Kunneth theorem allows us to identify this tensor product with $H^*(X \times Y) \simeq H^*((X \times Y)^\wedge)$, as desired.

**Corollary 8.** Let $A$ be a finitely generated abelian group and $n \geq 1$. Set $A^\vee = A \otimes \mathbb{Z}$. Then the canonical map $\hat{K}(A, n) \to K(A^\vee, n)$ is a homotopy equivalence.

**Proof.** Using Lemma 7 and the structure theory for finitely generated abelian groups, we can assume either that $A = \mathbb{Z}$ or that $A \simeq \mathbb{Z}/l^k\mathbb{Z}$, where $l$ is some prime number. In the first case, the desired result follows from Corollary 6. If $l = p$, then $K(A, n) = K(A^\vee, n)$ is $p$-finite and the result is obvious. If $l$ is distinct from $p$, then $K(A, n)$ has trivial cohomology (with coefficients in $F_p$), so that $\hat{K}(A, n)$ and $K(A^\vee, n)$ are both contractible.

**Lemma 9.** Suppose given a homotopy pullback square
\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Y
\end{array}
\]
of simply connected spaces, whose cohomology groups (with coefficients in \( \mathbb{F}_p \)) are finite dimensional in each degree. Then the induced square

\[
\begin{array}{ccc}
\tilde{X}' & \longrightarrow & \tilde{X} \\
\downarrow & & \downarrow \\
\tilde{Y}' & \longrightarrow & \tilde{Y}
\end{array}
\]

is a homotopy pullback diagram.

**Proof.** As before, it suffices to show that the diagram

\[
\begin{array}{ccc}
X' \wedge & \longrightarrow & X' \\
\downarrow & & \downarrow \\
Y' \wedge & \longrightarrow & Y'
\end{array}
\]

is a homotopy pullback diagram of \( p \)-profinite spaces, which is equivalent to the assertion that the diagram

\[
\begin{array}{c}
C^*(X') \leftarrow C^*(X) \\
\uparrow & \quad & \uparrow \\
C^*(Y') & \leftarrow & C^*(Y)
\end{array}
\]

is a homotopy pushout diagram of \( E_\infty \)-algebras over \( \mathbb{F}_p \). This is equivalent to the convergence of the cohomological Eilenberg-Moore spectral sequence; we proved this result in the case where all of the spaces involved were \( p \)-finite. However, our proof only used the finite dimensionality of cohomology groups and the nilpotence of the spaces involved; in particular, it remains valid when each space is simply connected and has cohomology of finite type.

We are now ready to prove our main result:

**Proof of Theorem 1.** Let \( X \) be a simply connected space whose homotopy groups are finitely generated. Then \( X \) has a Postnikov tower

\[
\ldots \rightarrow \tau_{\leq 3}X \rightarrow \tau_{\leq 2}X \rightarrow \tau_{\leq 1}X \simeq * ,
\]

where \( \tau_{\leq n}X \) is obtained from \( X \) by killing the homotopy groups of \( X \) above dimension \( n \). In particular, the map \( X \rightarrow \tau_{\leq n}X \) is highly connected if \( n \) is large, so that \( H^*X \simeq \lim_{\rightarrow} H^*\tau_{\leq n}X \). It follows that we have an equivalence of \( p \)-profinite spaces

\[
X^\vee \simeq \lim_{\rightarrow} (\tau_{\leq n}X)^\vee.
\]

Passing to the homotopy inverse limit, we get a homotopy equivalence

\[
\tilde{X} \simeq \lim_{\rightarrow} \tau_{\leq n}X.
\]

It will therefore suffice to prove the analogous result after replacing \( X \) by \( \tau_{\leq n}X \). We now proceed by induction on \( n \), using the existence of a homotopy pullback square

\[
\begin{array}{ccc}
\tau_{\leq n}X & \longrightarrow & * \\
\downarrow & \quad & \downarrow \\
\tau_{\leq n-1}X & \longrightarrow & K(\pi_nX, n + 1).
\end{array}
\]

The desired result now follows by combining the inductive hypothesis, Lemma 9, and Corollary 8. \qed
We conclude this section by giving a characterization of $\tilde{X}$ by a universal property. We first recall Bousfield’s notion of an $F_p$-local space.

**Definition 10.** A map $f : X \to Y$ of spaces is said to be an $F_p$-equivalence if the induced map on cohomology $H^*(Y) \to H^*(X)$ is an isomorphism.

A space $Z$ is said to be $F_p$-local if, for every $F_p$-equivalence $f : X \to Y$, the induced map $\text{Map}(Y,Z) \to \text{Map}(X,Z)$ is a homotopy equivalence.

**Example 11.** Every Eilenberg-MacLane space $K(F_p, n)$ is $F_p$-local (since the homotopy groups of the mapping space $\text{Map}(X, K(F_p, n))$ can be identified with cohomology groups of $X$ with coefficients in $F_p$).

It is clear that the collection of $F_p$-local spaces is closed under homotopy limits. Since every $p$-finite space $X$ can be built from Eilenberg-MacLane spaces $K(F_p, n)$ using finite homotopy limits, we conclude that $p$-finite spaces are $F_p$-local. It follows that any homotopy limit of $p$-finite spaces is again $F_p$-local. In particular, for any space $X$, the space $\tilde{X} = \lim X^\vee$ is $F_p$-local.

**Definition 12.** We say that a map of spaces $f : X \to X'$ exhibits $X'$ as an $F_p$-localization of $X$ if $f$ is an $F_p$-equivalence and $X'$ is $F_p$-local.

**Remark 13.** For any space $X$, there exists an $F_p$-localization $X'$ of $X$, and $X'$ is uniquely determined up to weak homotopy equivalence.

**Proposition 14.** Let $X$ be a simply connected space whose homotopy groups are finitely generated. Then the unit map $f : X \to \tilde{X}$ exhibits $\tilde{X}$ as an $F_p$-localization of $X$.

**Proof.** We have seen above that $\tilde{X}$ is $F_p$-local. It will therefore suffice to show that $f$ induces an isomorphism on cohomology with coefficients modulo $p$. Using the Serre spectral sequence repeatedly, we can reduce to the case where $X$ is an Eilenberg-MacLane space $K(A, n)$, where $A$ is a finitely generated abelian group. Then $\tilde{X} = K(A^\vee, n)$. We then have a fiber sequence

$$X \to \tilde{X} \to K(A^\vee/A, n).$$

Using the Serre spectral sequence again, it will suffice to show that the space $K(A^\vee/A, n)$ has trivial cohomology with coefficients in $F_p$. We can then invoke the following Lemma:

**Lemma 15.** Let $B$ be an abelian group such that multiplication by $p$ is an isomorphism from $B$ to itself, and let $n \geq 1$. Then $H_s K(B, n)$ vanishes for $s > 0$.

**Proof.** Since the functor $B \mapsto H_s K(B, n)$ commutes with filtered colimits, we may assume without loss of generality that $B$ is a finitely generated module over $\mathbb{Z}[1/p]$. Using the Eilenberg-Moore spectral sequence, we can assume $n = 1$. Using the structure theorem for finitely generated abelian groups and the Kunneth formula, we may assume either that $B = \mathbb{Z}[1/p]$ or that $B = \mathbb{Z}/l^k \mathbb{Z}$, where $l \neq p$. In the second case the result is clear: the homology of a finite group $G$ is always trivial at any prime which does not divide the order $|G|$. In the first case, $K(B, 1)$ is the homotopy colimit of the sequence

$$S^1 \overset{p}{\to} S^1 \overset{p}{\to} S^1 \to \ldots,$$

so we have $H_* K(B, 1) \simeq \lim H_* S^1$ and the result follows by inspection.

**Remark 16.** For a general space $X$, the unit map $X \to \tilde{X}$ need not induce an isomorphism on $F_p$-cohomology, so that $\tilde{X}$ need not be an $F_p$-localization of $X$. 

\[\square\]