SYMPLECTIC GEOMETRY, LECTURE 12

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1. Existence of Almost-Complex Structures

Let \((M, \omega)\) be a symplectic manifold. If \(J\) is a compatible almost-complex structure, we obtain invariants \(c_2(TM, J) \in H^2(M, \mathbb{Z})\) of the deformation equivalence class of \((M, \omega)\).

**Remark.** There exist 4-manifolds \((M^4, \omega_1), (M^4, \omega_2)\) s.t. \(c_1(TM, \omega_1) \neq c_1(TM, \omega_2)\).

We can use this to obtain an obstruction to the existence of an almost-complex structure on a 4-manifold: note that we have two Chern classes \(c_1(TM, J) \in H^2(M, \mathbb{Z})\) and \(c_2(TM, J) = e(TM) \in H^4(M, \mathbb{Z}) \cong \mathbb{Z}\) if \(M^4\) is closed, compact. Then the class

\[
(1 + c_1 + c_2)(1 - c_1 + c_2) - 1 = -c_2^2 + 2c_2 = c_2(TM \oplus TM, J \oplus J) = c_2(TM \otimes_{\mathbb{R}} \mathbb{C}, i)
\]

is independent of \(J\).

More generally, for \(E\) a real vector space with complex structure \(J\), we have an equivalence \((E \otimes_{\mathbb{R}} \mathbb{C}, i) \cong E \oplus E = (E, J) \oplus (E, -J)\). Indeed, \(J\) extends \(\mathbb{C}\)-linearly to an almost complex structure \(J_\mathbb{C}\) which is diagonalizable with eigenvalues \(\pm i\). Applying this to vector bundles, we obtain the **Pontrjagin classes**

\[
p_1(TM) = -c_2(TM \otimes_{\mathbb{R}} \mathbb{C}) \in H^4(M, \mathbb{Z}) \cong \mathbb{Z}
\]

for a 4-manifold \(M\).

**Theorem 1.** \(p_1(TM) \cdot [M] = 3\sigma(M)\), where \(\sigma(M)\) is the signature of \(M\) (the difference between the number of positive and negative eigenvalues of the intersection product \(Q : H_2(M) \otimes H_2(M) \rightarrow \mathbb{Z}, [A] \otimes [B] \mapsto [A \cap B]\) dual to the cup product on \(H^2\)).

**Corollary 1.** \(c_1^2 \cdot [M] = 2\chi(M) + 3\sigma(M)\).

**Remark.** Under the map \(H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z}/2\mathbb{Z})\), the Chern class \(c_1(TM, J)\) gets sent to the **Stieffel-Whitney class** \(w_2(TM)\). This means that

\[
c_1(TM) \cdot [A] \equiv Q([A], [A]) \mod 2 \forall [A] \in H_2(M, \mathbb{Z})
\]

**Theorem 2.** \(\exists\) an almost complex structure \(J\) on \(M^k\) s.t. \(\alpha = c_1(TM, J) \in H^2(M, \mathbb{Z})\) iff \(\alpha\) satisfies

\[
\alpha^2 \cdot [M] = 2\chi + 3\sigma \text{ and } \alpha \cdot [A] \equiv Q([A], [A]) \mod 2 \forall [A] \in H_2(M, \mathbb{Z})
\]

**Examples:**

- On \(S^4\), if \(J\) were an almost complex structure, then \(c_1(TS^4, J) \in H^2(S^4) = 0\). However, \(\chi(S^4) = 2\) and \(\sigma(S^4) = 0\), so \(2 \cdot 2 + 3 \cdot 0\) cannot be \(c_1^2\), and thus there is no almost complex structure.

- On \(\mathbb{CP}^2\), we have \(H_2(\mathbb{CP}^2, \mathbb{Z}) = \mathbb{Z}\) generated by \([\mathbb{CP}^1]\) with intersection product \(Q([\mathbb{CP}^1], [\mathbb{CP}^1]) = 1\) (the number of lines in the intersection of two planes in \(\mathbb{C}^3\)). By Mayer-Vietoris, \(H_2(\mathbb{CP}^2 \# \mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z}^2\) has intersection product \(Q = I_{2 \times 2} \implies \sigma = 2\) and Euler characteristic \(\chi = 4\). Now, assume \(c_1(TM, J) = (a, b) \in H_2(M, \mathbb{Z})\): if there were an almost complex structure,

\[
a^2 + b^2 = c_1^2 = 2\chi + 3\sigma = 14
\]

which is impossible.
2. Types and Splittings

Let \((M, J)\) be an almost complex structure, \(J\) extended \(\mathbb{C}\)-linearly to \(TM \otimes \mathbb{C} = TM^{1,0} \oplus TM^{0,1}\) (with the decomposition being into \(+i\) and \(-i\) eigenspaces). Here, \(TM^{1,0} = \{v-iJv | v \in TM\}\) is the set of holomorphic tangent vectors and \(TM^{0,1} = \{v+iJv, v \in TM\}\) is the set of anti-holomorphic tangent vectors. For instance, on \(\mathbb{C}^n\), this gives

\[
\frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) = \frac{\partial}{\partial x_j}, \quad \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) = \frac{\partial}{\partial \bar{z}_j}\]

respectively. More generally, we have induced real isomorphisms

\[
\pi^{1,0} : TM \to TM^{1,0}, \quad v \mapsto \nu^{1,0} = \frac{1}{2}(v - iJv), \quad \pi^{0,1} : TM \to TM^{0,1}, \quad v \mapsto \nu^{0,1} = \frac{1}{2}(v + iJv)
\]

Then \((\nu^{1,0}), (\nu^{0,1})\) is an almost-complex bundle.

Similarly, the complexified cotangent bundle decomposes as \(T^*M^{1,0} = \{\eta \in T^*M \otimes \mathbb{C} | \eta(Jv) = i\eta(v)\}\), \(T^*M^{0,1} = \{\eta \in T^*M \otimes \mathbb{C} | \eta(Jv) = -i\eta(v)\}\), with maps from the original cotangent bundle given by

\[
\eta \mapsto \eta^{1,0} = \frac{1}{2}(\eta - i(\eta \circ J)), \quad \eta \mapsto \eta^{0,1} = \frac{1}{2}(\eta + i(\eta \circ J)) = \frac{1}{2}(\eta - iJ^*\eta)
\]

For \(\mathbb{C}^n\), we find that

\[
J^*dx_i = dy_i, J^*dy_i = -dx_i \implies dx_j + i dy_j = dz_j, (\ast \nu C^1, 0), dx_j - idy_j = d\overline{z}_j \in (\ast \nu C^n, 0)\]

More generally, on a complex manifold, in holomorphic local coordinates, we have \(T^*M^{1,0} = \text{Span}(dz_j)\). Note also that \(T^*M^{1,0}\) pairs with \(TM^{1,0}\) trivially.

2.1. Differential forms. \(\Omega^k\) splits into forms of type \((p, q)\), \(p + q = k\), with

\[
\wedge^{p,q}T^*M = (\wedge^p T^*M^{1,0}) \otimes (\wedge^q T^*M^{0,1}) = \bigoplus_{p+q=k} \wedge^{p,q}T^*M
\]

**Definition 1.** For \(\alpha \in \Omega^{p,q}(M)\), \(\partial\alpha = (da)^{p+1,q} \in \Omega^{p+1,q}\) and \(\bar{\partial}\alpha = (da)^{p,q+1} \in \Omega^{p,q+1}\).

In general,

\[
d\alpha = (da)_{p+q+1} + (da)_{p+q} + \cdots + (da)_{0,p+q+1}
\]

For a function, we have \(df = \partial f + \bar{\partial} f\). Now, say \(f : M \to \mathbb{C}\) is \(J\)-holomorphic if \(\bar{\partial} f = 0 \Leftrightarrow df \in \Omega^{1,0} \Leftrightarrow df(Jv) = idf(v)\).

2.2. Dolbeault cohomology. Assume \(d\) maps \(\Omega^{p,q} \to \Omega^{p+1,q} \oplus \Omega^{p,q+1}\), i.e. \(d = \partial + \bar{\partial}\). On \(\mathbb{C}^n\), for instance, we have

\[
\partial(\alpha_1 dz_1 \wedge \cdots \wedge dz_p \wedge \alpha_2 dz_1 \wedge \cdots \wedge dz_q) = \sum_k \frac{\partial z_k}{\partial x_k} \left( dz_k \wedge dz_1 \wedge \cdots \wedge dz_p \wedge dz_1 \wedge \cdots \wedge dz_q \right)
\]

\[
\bar{\partial}(\alpha_1 \overline{dz_1} \wedge \cdots \wedge \overline{dz_p} \wedge \alpha_2 \overline{dz_1} \wedge \cdots \wedge \overline{dz_q}) = \sum_k \frac{\partial z_k}{\partial \bar{z}_k} \left( dz_k \wedge \overline{dz_1} \wedge \cdots \wedge \overline{dz_p} \wedge \overline{dz_1} \wedge \cdots \wedge \overline{dz_q} \right)
\]

Then, \(v \in \Omega^{p,q}, 0 = \partial^2 = \partial \bar{\partial} + \bar{\partial} \partial = \bar{\partial} \partial = 0\). Since \(\bar{\partial}^2 = 0\), we obtain a complex \(0 \to \Omega^{p,0} \to \Omega^{p,1} \to \cdots\).

**Definition 2.** The Dolbeault cohomology of \(M\) is

\[
H^{p,q}(M) = \frac{\operatorname{Ker}(\partial : \Omega^{p,q} \to \Omega^{p,q+1})}{\operatorname{Im}(\bar{\partial} : \Omega^{p,q-1} \to \Omega^{p,q})}
\]

In general, this is not finite-dimensional. We’ll see that on a compact Kähler manifold, i.e. a manifold with compatible symplectic and complex structures, \(H^k(M, \mathbb{C}) = \bigoplus_{p+q = k} H^{p,q}(M)\).
2.3. **Integrability.** Let \((M, J)\) be a manifold with almost-complex structure.

**Definition 3.** The Nijenhuis tensor is the map \(N(u, v) = [Ju, Jv] - J[u, Jv] - J[Ju, v] - [u, v]\) for \(u, v\) vector fields on \(M\).

In fact, \(N(u, v) = -8\text{Re}(u^{1,0}, v^{1,0})^{0,1}\).