16 Lecture 21-23 (Notes: K. Venkatram)

16.1 Linear Algebra

We define a category \( \mathcal{H} \) whose objects are pairs \((E, g)\) (sometimes denoted \( E \) for brevity), where \( E \) is a finite dimensional vector space over \( \mathbb{R} \) and \( g \) is a nondegenerate symmetric bilinear form on \( E \) with signature 0, and whose morphisms are maximal isotropies \( L \subset E \times F \). Here, \( E \mapsto \overline{E} = (E, -g) \) is the natural involution, and \( E \times F = (E \times F, g_E + g_F) \) is the natural product structure. Composition is done by composition of relations, i.e.

\[
E \xrightarrow{L} F \xrightarrow{M} G, M \circ L = \{(e, g) \in E \times G | \exists f \in F \text{ s.t. } (e, f) \in L, (f, g) \in M\}.
\]

**Proposition 11.** \( M \circ L \) is a morphism in \( \mathcal{H} \).

**Proof.** \( L : L \times M \subset \overline{E} \times F \times \overline{F} \times G = W \) is maximally isotropic. \( \mathcal{C} = E \times \Delta_F \times G \), where \( \Delta_F = \{(f, f) | f \in F\} \), is coisotropic, i.e. \( \mathcal{C}^\perp = \Delta_F \subset \mathcal{C} \). Thus, we get an induced bilinear form on \( \mathcal{C}^\perp / \mathcal{C} = \overline{E} \times G \). \( \mathcal{C} \cap L + \mathcal{C}^\perp \) is maximal isotropic in \( W \), so

\[
(\mathcal{C} \cap L + \mathcal{C}^\perp)^\perp = (\mathcal{C}^\perp + \mathcal{L}^\perp) \cap \mathcal{C} = \mathcal{C}^\perp + \mathcal{L} \cap \mathcal{C} \tag{125}
\]

Thus, \( \mathcal{C} \cap L + \mathcal{C}^\perp / \mathcal{C}^\perp = M \circ L \subset \mathcal{C} / \mathcal{C}^\perp = \overline{E} \times G \) is maximally isotropic. \( \square \)

**Remark.** This category is the symmetric version of the Weinstein’s symplectic category \( \zeta \) where \( \text{Ob}(\zeta) = (E, \omega) \) and morphisms are given by Lagrangians. Thus, it is the ”odd” version or parity reversal of \( \zeta \).

A particular case of a morphism \( E \to F \) is the graph of an orthogonal morphism.

**Problem.** Show that \( L : E \to F \) is epi \( \iff \pi_F(L) = F \), mono \( \iff \pi_E(L) = E \), and iso \( \iff L \) is orthogonal iso \( E \to F \).

So for \( \text{dim } E = 2n, O(n, n) \subset \text{Hom}(E, E) \) are isos. But \( \text{Hom}(E, E) \cong O(2n) \) as a space since we can choose a positive definite \( C_+ \) and then any \( L \in O(2n) \). This implies that \( \text{Hom}(E, E) \) is a monoid compactifying the group \( O(E) \).

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16.1.1 Doubling Functor

Now, there is a nature "Double" functor \( \mathcal{D} : \text{Vect} \to \mathcal{H} \) which maps \( V \to V \oplus V^* \) and 
\[ \{ f : V \to M \} \mapsto \{ (v + f^* \eta, f_* v + \eta) \in V \oplus V^* \times W \oplus W^* | v \in V, \eta \in W^* \} \]. Note that 
\( \mathcal{D}f \subset \mathcal{D}V \times \mathcal{D}W \) and \( \dim \mathcal{D}f = \dim V + \dim W \).

\[
\langle (v + f^* \eta, f_* v + \eta), (v + f^* \xi, f_* v + \xi) \rangle = -f^* \eta(v) + \eta(f_* v) = 0
\]

(126)

**Problem.** Prove that \( \mathcal{D} \) is a functor, i.e. \( \mathcal{D}(f \circ g) = \mathcal{D}f \circ \mathcal{D}g \).

Note that \( \mathcal{H} \) has a duality functor \( L \in \text{Hom}(E, F) \implies L^* \in \text{Hom}(F, E) \), where \( L^* = \{(f, e) | (e, f) \in L \} \).

**Problem.** Show that \( \mathcal{D}(f^*) = (\mathcal{D}f)^* \).

**Problem.** Prove that \( \mathcal{D} \) preserves epis and monos.

16.1.2 Maps Induced by Morphisms

A morphism \( L \in \text{Hom}(E, F) \) induces maps \( L \circ - : \text{Hom}(X, E) \to \text{Hom}(X, F) : L^* \circ - \). A special case is \( X = \{0\} \), in which \( \text{Hom}(0, E) = \text{Dir}(E) \), so \( L \in \text{Hom}(E, F) \) induces maps \( L_a : \text{Dir}(E) \to \text{Dir}(F) : L^* \). If \( L \)
is mono or epic, so is \( L_a \). This recovers the pushforward and pullback of Dirac structures: for \( f : V \to W \) a linear map, \( \mathcal{D}f : \mathcal{D}V \to \mathcal{D}W \) a morphism we obtain maps \( \mathcal{D}f_a : \text{Dir}(V) \to \text{Dir}(W) : \mathcal{D}f^* \). As observed
earlier, any Dirac \( L \subset V \oplus V^* \) with \( \pi_V(L) = M \subset V \) can be written as \( L(M, B) \), \( B \in \Lambda^2 M^* \), i.e. \( L = j_* \Gamma_B \) for \( j : M \to V \) the embedding and a unique \( B \). That is, \( L = j_* e^B M \).

**Example.** Given \( f : V \to W \) a linear map, \( \mathcal{D}f \subset \mathcal{D}V \times \mathcal{D}W = \mathcal{D}(V \oplus W^*) \), and \( \mathcal{D}f = ((f, f^* \eta), (f_* v, \eta) \cdots ) \), hence \( \pi_{V \oplus W} \circ \mathcal{D}f = V \oplus W^* \) is onto. Therefore, \( \mathcal{D}f = e^B(V \oplus W^*) \), and in fact \( B = f \in V^* \oplus W \subset \Lambda^2 (V \oplus W^*)^* \).

16.1.3 Factorization of Morphisms \( L : \mathcal{D}V \to \mathcal{D}(W) \)

Let \( L \in \text{Hom}(\mathcal{D}V, \mathcal{D}W) \), \( L \subset \mathcal{D}V \times \mathcal{D}W \cong \mathcal{D}(V \oplus W) \). Then \( L = j_* e^F M \), for \( M = \pi_{V \oplus W} L \subset V \oplus W \). Let \( \phi : M \to V, \psi : M \to W \) be the natural projections.

**Theorem 13.** \( L = \mathcal{D} \psi_* \circ e^F \circ \mathcal{D} \phi^* \).

**Proof.** (Exercise) \( \square \)

**Corollary 10.** \( L \) is an isomorphism \( \iff \phi, \psi \) are surjective and \( F \) determines a nondegenerate pairing \( \text{Ker } \phi \times \text{Ker } \psi \to \mathbb{R} \).

Therefore, an orthogonal map \( V \oplus V^* \to W \oplus W^* \) can be viewed as a subspace \( M \subset V \times W, F \in \Lambda^2 M^* \).

16.2 T-duality

The basic idea of T-duality is as follows: let \( S^1 \to P \to \pi B \) be a principal \( S^1 \) bundle, i.e. a spacetime with geometry, with an invariant 3-form flux \( H \in \Omega_3^1(P) S^1 \) and an integral \( |H| \in H^3(P, Z) \), i.e. coming from a gerbe with connection. Then we are going to produce a new "dual" spacetime with "isomorphic quantized field theory" (in this case, a sigma model). Specifically, let \( \hat{P} \) be a new \( S^1 \) bundle over \( B \) so that \( c_1(\hat{P}) = \pi_* (H) \in H^2(B, Z) \), and choose \( \hat{H} \in H^3(\hat{P}, Z) \) s.t. \( \hat{\pi}^* \hat{H} = c_1(\hat{P}) \). More specifically, choose a connection \( \theta \in \Omega^1(P) \) (i.e. \( L_{\partial \theta} \theta = 0, i_{\partial \theta} = 1/2 \pi \)) so \( d\theta = F \in \Omega^2(B) \) is integral and \( [F] = c_1(P) \). Then \( H = \hat{F} \wedge \theta + h \) for some \( \hat{F} \in \Omega^2(B) \) integral and \( H \in \Omega^3(B) \). Now, \( [\hat{F}] \in H^2(B, Z) \) defines a new principal \( S^1 \) bundle \( \hat{P} \). Choose a connection \( \hat{\theta} \) on \( \hat{P} \) so that \( d\hat{\theta} = \hat{F} \). Then define \( \hat{H} = F \wedge \hat{\theta} + h \), so that \( \int \hat{H} = F \) and \( \int H = F \).
Example. Let $S^1 \times S^2 \to S^2$ be the trivial $S^1$-bundle, with $H = v_1 \wedge v_2$. Then $v_2 = \int_{S^1} H = c_1(S^3 \to S^2)$, so the $T$-dual is the pair $S^3, 0$. Our original space has trivial topology and nontrivial flux, while the new space has nontrivial topology and trivial flux.

Remark. In physics, $T$-dual spaces have the same quantum physics, hence the same $D$-branes and twisted $K$-theory.

**Theorem 14 (BHM).** We have an isomorphism $K_H^*(P) \cong K_H^{*+1}(\tilde{P})$.

Next, let $P \times B \tilde{P} = \{(p, \tilde{p}) | \pi(p) = \tilde{\pi}(\tilde{p}) \} \subset P \times \tilde{P}$ be the correspondence space, $\phi, \psi$ the two projections. Then $\phi^* H - \psi^* \tilde{H} = F \wedge \theta - F \wedge \tilde{\theta} = -d(\phi^* \theta \wedge \psi^* \tilde{\theta})$.

**Definition 23.** A $T$-duality between $S^1$-bundles $(P, H)$ and $(\tilde{P}, \tilde{H})$ over $B$ is a 2-form $F \in \Omega^2(P \times_B \tilde{P})^{S^1 \times S^1}$ s.t. $\phi^* H - \psi^* \tilde{H} = dF$ and $F$ determines a nondegenerate pairing $\text{Ker} \phi \times \text{Ker} \psi \to \mathbb{R}$.

In fact, $T$-duality can be expressed, therefore, as an orthogonal isomorphism

$$\frac{(T_p \oplus T_p^*, H)}{S^1} \to \frac{L(P \times_B \tilde{P}, F)}{T_p \oplus T_p^*, \tilde{H}}/S^1$$

(127)

though of as bundles over $B$ (or just $S^1$-invariant sections on $P, \tilde{P}$). This map sends $H$-twisted bracket to $\tilde{H}$-twisted bracket, via

$$\Omega^*(P)^{S^1} \ni \rho \mapsto \tau(\rho) = \psi_* e^F \wedge \phi^* \rho = \int_{S^1} e^F \wedge \phi^* \rho \in \Omega^*(\tilde{P})^{S^1}$$

(128)

Since $d(e^F \rho) = e^F (d\rho + (H - \tilde{H})\rho)$, we find that $dH(e^F \rho) = e^F dH \rho$ and $\tau(dH \rho) = d\tilde{H}\tau(\rho)$ as desired.

Overall, a $T$-duality $F : (P, H) \to (\tilde{P}, \tilde{H})$ implies an isomorphism $(T_p \oplus T_p^*, H)/S^1 \to L(P \times_B \tilde{P}, F)(T_p \oplus T_p^*, \tilde{H})/S^1$ as Courant algebroid, and thus any $S^1$-invariant generalized structure may be transported from $(P, H)$ to $(\tilde{P}, \tilde{H})$.

**Example.**

1. $T_p \subset (T_p \oplus T_p^*, H)$ is a Dirac structure $\implies$ $T$-dual is

$$\tau(\xi + \theta) = \xi - \tilde{\theta} = T^* B + \langle \partial_\theta \rangle = \Delta \oplus \text{Ann} \Delta$$

(129)

for $\delta = \langle \partial_\theta \rangle$

2. The induced map on twisted cohomology $H^*_H(P) \cong H^{*+1}_H(\tilde{P})$ is an isomorphism.

3. Where does $\tau$ take the subspace $C_+ = \Gamma_{g+b} \subset T^* \oplus T$? In $TP = TB \oplus 1$, decompose

$$g = g_0 \theta \circ \theta + g_1 \circ \theta + g_2, b = b_1 \circ \theta + b_2$$

for $g_i, b_i$ basic. Then

$$C_+ = \Gamma_{g+b} = \{ x + f \partial_\theta + (i_x g_2 + fg_1 + i_x b_2 - fb_1) + (g_1(x) + fg_0 + b_1(x))\theta \}$$

(130)

which is mapped via $\tau$ to

$$\Gamma_{\tilde{g} + \tilde{b}} = \{ x + (g_1(x) + fg_0 + b_1(x))\tilde{\partial}_\theta + (i_x g_1 + fg_1 + i_x b_2 - fb_1) + f\tilde{\theta} \}$$

(131)

where

$$\begin{cases}
\tilde{g} = \frac{1}{g_0} \tilde{g} \circ \tilde{\theta} - \frac{b_1}{g_0} \circ \tilde{\theta} + g_2 + \frac{1}{g_0}(b_1 \circ b_1 - g_1 \circ g_1) \\
\tilde{b} = \frac{g_1}{g_0} \wedge \tilde{\theta} + b_2 + \frac{g_1 \wedge b_1}{g_0}
\end{cases}$$

(132)

These are called "Buscher rules".

4. Elliptic Curves:

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