7 Lecture 7 (Notes: N. Rosenblyum)

7.1 Exact Courant Algebroids

Recall that a Courant algebroid is given by the diagram of bundles

\[ \xymatrix{ E \ar[r]^\pi & T \ar[dl]_\pi \ar[dr]^\pi \ar[d]_M \ar[ddd] \ar[ddd] & \text{where } \pi \text{ is called the "anchor" along with a bracket }, [ , ] \text{ and a nondegenerate bilinear form } \langle , \rangle \text{ such that} \]

- \[ \pi[a,b] = [\pi a, \pi b] \]
- The Jacobi identity is zero
- \[ [a,f b] = f [a,b] + ((\pi a)f)b \]
- \[ [a,b] = \frac{1}{2}\pi^* d\langle a, a \rangle \]
- \[ \pi a(b,c) = \langle [a,b], c \rangle + \langle b, [a,c] \rangle \]

A Courant algebroid is exact if the sequence

\[ 0 \longrightarrow T^* \xrightarrow{\pi^*} E \xrightarrow{\pi} T \longrightarrow 0 \]

is exact (note that \( \pi \circ \pi^* \) is always 0).

Remarks: For an exact Courant algebroid, we have:

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1. The inclusion $T^* \subset E$ is automatically isotropic because for $\xi, \eta \in T^*$,
\[
\langle \pi^* \xi, \pi^* \eta \rangle = \xi(\pi^* \eta) = 0
\]
since $\langle \pi^* \xi, a \rangle = \xi(\pi a)$.

2. The bracket $[\cdot, \cdot]_{T^*} = 0$: for $s, t \in C^\infty(E), f \in C^\infty(M)$,
\[
\mathcal{D} = \pi^* d : C^\infty(M) \to C^\infty(E)
\]

Now,
\[
\langle [s, \mathcal{D} f], t \rangle = \pi s(\mathcal{D} f, t) - \langle \mathcal{D} f, [s, t] \rangle = \pi s(\pi t(f)) - \pi [s, t](f) = \pi t(\pi s(f)) = \langle \mathcal{D} [\mathcal{D} f, s], f \rangle
\]

Thus, $[s, \mathcal{D} f] = \mathcal{D}(s, \mathcal{D} f)$. We also have, $[\mathcal{D} f, s] + [s, \mathcal{D} f] = \mathcal{D}(\mathcal{D} f, s)$ and therefore $[\mathcal{D} f, s] = 0$.

We need to show that $[f dx^i, g dx^j] = 0$. But have $[dx^i, dx^j] = 0$ and
\[
[a, fb] = f[a, b] + ((\pi a)f)b, \quad [ga, b] = g[a, b] - ((\pi b)g)a + 2(a, b)dg.
\]

### 7.2 Severa’s Classification of Exact Courant Algebroids

We can choose an isotropic splitting
\[
0 \xrightarrow{\pi} T^* \xrightarrow{\pi^*} E \xrightarrow{\pi} T \xrightarrow{\pi} 0
\]
i.e. $\langle sX, sY \rangle = 0$ for all $X, Y \in T$. We then have $E \cong T \oplus T^*$ and we can transport the Courant structure to $T \oplus T^*$: for $X, Y \in T$ and $\xi, \eta \in T^*$,
\[
\langle X + \xi, Y + \eta \rangle = \langle sX + \pi^* \xi, sY + \pi^* \eta \rangle = \xi(\pi sY) + \eta(\pi sX) = \xi(Y) + \eta(X)
\]
since $\langle sX, sY \rangle = 0$. Also,
\[
[X + \xi, Y + \eta] = [sX + \pi^* \xi, sY + \pi^* \eta] = [sX, sY] + [sX, \pi^* \eta] + [\pi^* \xi, sY]
\]

We have that the second term is given by
\[
\pi [sX, \pi^* \eta] = [\pi sX, \pi \pi^* \eta] = 0
\]
and therefore, $[sX, \pi^* \eta] \in \Omega^1$. Further,
\[
[sX, \pi^* \eta](Z) = \langle [sX, \pi^* \eta], sZ \rangle = X(\pi^* \eta, sZ) - \langle \pi^* \eta, [sX, sZ] \rangle = X \eta(Z) - \eta([X, Z]) = i_z L_X \eta
\]
and so $[sX, \pi^* \eta] = L_X \eta$.

Now, the third term is given by
\[
\langle [\pi^* \xi, sY], sZ \rangle = -\langle sY, \pi^* \xi \rangle + \mathcal{D} \langle sY, \pi^* \xi \rangle, sZ \rangle = -(L_Y \xi)(Z) + i_z d_Y \xi = -(i_Y d_\xi)(Z)
\]
and so $[\pi^* \xi, sY] = -i_Y d_\xi$.

For the first term, we have no reason to believe that $[sX, sY] = [X, Y]$ We do have that $\pi [sX, sY] = [X, Y]_{Lie}$. Now, let $H(X, Y) = s^*[sX, sY]$. We then have,
1. $H$ is $C^\infty$-linear and skew in $X,Y$:

$$H(X,fY) = f s^*[sX,sY] + s^*(X(f)sY) = f s^*[sX,sY],$$

and

$$H(fX,Y) = s^*[f sX,sY] = fH(X,Y) - s^*((Yf)sX) + 2(sX,sY)(df) = f H(X,Y).$$

Furthermore,

$$[sX,sY] + [sY,sX] = \pi^*d(sX,sY).$$

2. $H(X,Y)(Z)$ is totally symmetric in $X,Y,Z$:

$$H(X,Y)(Z) = \langle[sX,sY],sZ\rangle_E = X(sY,sZ) - \langle sY,[sX,sZ]\rangle$$

So, we have $[sX,sY] = [X,Y] - i_Y i_X H$ for $H \in \Omega^3(M)$.

**Problem.** Show that $[[a,b],c] = [a,[b,c]] - [b,[a,c]] + i_\pi j_\pi j_\pi dH$ and so $Jac = 0$ if and only if $dH = 0$.

Thus, we have that the only parameter specifying the Courant bracket is a closed three form $H \in \Omega^3(M)$. We will see that when $[H]/2\pi \in H^3(M,\mathbb{Z})$, $E$ is associated to an $S^1$-gerbe.

Now, let’s consider how $H$ changes when we change the splitting. Suppose that we have two section $s_1, s_2 : T \to E$. Then, we have that $\pi(s_1 - s_2) = 0$. So consider $B = s_1 - s_2 : T \to T^*$. In the $s_1$ splitting, we have for $x \in T$, $s_2(x) = (x + (s_2 - s_1))x$. Since the $s_i$ are isotropic splittings, we have that $(s_2 - s_1)(x)(x) = 0$. Thus we have, $B \in C^\infty(T)$.

Now, in the $s_1$ splitting we have,

$$[X+iXB,Y+iYB]_H = [X,Y] + L_X i_Y B - i_Y L_X B + i_Y i_X H = [X,Y] + i_{[X,Y]} B + i_Y i_X (H + dB)$$

In particular, in the $s_2$ splitting $H$ changes by $dB$. Thus, we have that $[H] \in H^3(M,\mathbb{R})$ classifies the exact Courant algebroid up to isomorphism. The above bracket is also a derived bracket. Before, we had that

$$[a,b]_C \cdot \varphi = [[d,a],b] \varphi.$$ 

Now, replace $d$ with $d_H = d + H \wedge$. We clearly have that $d^2_H = (dH) \wedge = 0$ since $dH = 0$. Note that $d_H$ is not of degree one and is not a derivation but it is odd. The cohomology of $d_H$ is called $H$-twisted deRham cohomology. In simple cases (e.g. when $H$ is formal in the sense of rational homotopy theory), we have

$$H^*(H^{ev/odd}(M), e[H]) = H^{ev/odd}_d(M)$$

where $e_H = H \wedge$.

Now, $[a,b]_H \cdot \varphi = [[d_H,a],b] \varphi$. Indeed, for $B \in \Omega^2$, we have $\varphi \mapsto e^B \varphi$ and $e^{-B}(d + H \wedge) e^B = e^{-B} d e^B + e^{-B} H e^B = d_{H + dB}$, and so $e^B [e^{-B} \cdot , e^B]_H = [\ , ]_{H + dB}$ In particular, if $B \in \Omega^2_T$, then $e^B$ is a symmetry of the Courant bracket.

This phenomena is somewhat unusual because for the ordinary Lie bracket, the only symmetries are given by diffeomorphisms of the underlying manifold. More specifically, a symmetry of the Lie bracket on $C^\infty(T)$ is a diagram

$$\begin{array}{ccc} 
T & \xrightarrow{\Phi} & T \\
\downarrow & & \downarrow \\
M & \xrightarrow{\phi} & M 
\end{array}$$

such that $\phi$ is a diffeomorphism and $[\Phi,\Phi] = \Phi[\cdot,\cdot]$. 

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Claim 1. $\text{Sym}[\ , ]_{\text{Lie}} = \{(\phi_*, \phi), \phi \in \text{Diff}(M)\}$. 

Proof. Given $(\Phi, \phi) \in \text{Sym}[\ , ]_{\text{Lie}}$, consider $G : \Phi \phi^{-1}$. Then $G$ covers the identity map on $M$ and we have $fG[X, Y] - ((Yf)GX = G[fX, Y] = f[GX, GY] - (GY)fGX$ and so $Yf = (GY)(f)$ for all $Y, f$ and so $G = 1$.

Let’s now consider the question of what all the symmetries of the Courant bracket $[\ , ]_C$ are. Once again, we have a diagram

$$
E \xrightarrow{\Phi} E \\
\downarrow \quad \downarrow \\
M \xrightarrow{\phi} M
$$

where $E \simeq T \oplus T^*$ such that

1. $\phi^* \langle \Phi, \Phi \rangle = \langle \cdot, \cdot \rangle$
2. $[\Phi, \Phi] = \Phi[\cdot, \cdot]$
3. $\pi \circ \Phi = \phi_* \circ \pi$.

Suppose that $\phi \in \text{Diff}(M)$. Then on $T \oplus T^*$, $\phi_*$ is given by

$$
\phi_* = \left( \begin{array}{c}
\phi_* \\
(\phi^*)^{-1}
\end{array} \right)
$$

and so we have $\phi_*(X + \xi) = \phi_*X + (\phi^*)^{-1}\xi$ and

$$
\phi_*^{-1}[(\phi_*X + (\phi^*)^{-1}\xi, \phi_*Y + (\phi^*)^{-1}\eta)]_H = [X + \xi, Y + \eta]_{\phi^*H}
$$

since $\phi_*^{-1}(i_{\phi_*Y}i_{\phi_*X}H)(Z) = i_{\phi_*Z}i_{\phi_*Y}i_{\phi_*X}H = \phi^*H(X, Y, Z)$. In particular, this does not give a symmetry unless $\phi^*H = H$.

Now, consider a $B$-field transform. Since $e^B[e^{-B}, e^{-B}]_H = [\cdot, \cdot]_{H+dB}$, this is not a symmetry unless $dB = 0$. Now we can combine these to generate the symmetries:

$$
[\phi_*e^B, \phi_*e^B] = \phi_*e^B[\cdot, \cdot]_{\phi^*H+dB}
$$

and so $\phi_*e^B \in \text{Sym}E$ iff $H - \phi^*H = dB$. It turns out that these are all the symmetries.

Theorem 5. The above are all the symmetries of an exact Courant algebroid. In particular, we have a short exact sequence

$$
0 \rightarrow \Omega^2_{cl} \rightarrow \text{Sym}(E) \rightarrow \text{Diff}_{[H]} \rightarrow 0
$$

where $\text{Diff}_{[H]}$ is the subgroup of diffeomorphisms of $M$ preserving the cohomology class $[H]$. 

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