18.969 Topics in Geometry: Mirror Symmetry
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1. The Quintic (contd.)

To recall where we were, we had

\( \{ (x_0 : \cdots : x_4) \in \mathbb{P}^4 \mid f_\psi = \sum_{0}^{4} x_i^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0 \} \)

with

\( G = \{(a_0, \ldots, a_4) \in (\mathbb{Z}/5\mathbb{Z})^5 \mid \sum a_i = 0 \}/\{(a, a, a, a, a)\} \cong (\mathbb{Z}/5\mathbb{Z})^3 \)

acting by diagonal multiplication \( x_i \mapsto x_i \xi^{a_i}, \xi = e^{2\pi i/5} \). We obtained a crepant resolution \( \tilde{X}_\psi \) of \( X_\psi / G \). This family has a LCSL point at \( z = (5\psi)^{-5} \to 0 \). There was a volume form \( \Omega_\psi \) on \( \tilde{X}_\psi \) induced by the \( G \)-invariant volume form \( \Omega_\psi \) on \( X_\psi \) by pullback via \( \pi : \tilde{X}_\psi \to X_\psi / G \). We computed its period on the 3-torus

\[ T_0 = \{(x_0 : \cdots : x_4) \mid x_4 = 1 | x_0 | = | x_1 | = | x_2 | = \delta, | x_3 | \ll 1 \} \]

(or, on the mirror, \( \tilde{T}_0 \subset \tilde{X}_\psi \)) to be

\[ \int_{T_0} \Omega_\psi = -(2\pi i)^3 \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^3 (5\psi)^{5n}} \]

In terms of \( z = (5\psi)^{-5} \), the period is proportional to

\[ \phi_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^3} z^n \]

Setting \( \Theta = z \frac{d}{dz} : \Theta(\sum c_n z^n) = \sum n c_n z^n \), we obtained the Picard-Fuchs equation

\[ \theta^4 \phi_0 = 5z(5\Theta + 1)(5\Theta + 2)(5\Theta + 3)(5\Theta + 4) \phi_0 \]

**Proposition 1.** All periods \( \int \Omega_\psi \) satisfy this equation.

Note that all period satisfy some 4th order differential equation: \( H^3(\tilde{X}_\psi, \mathbb{C}) \) is 4-dimensional, so \([ \Omega_\psi], \frac{d}{d\psi} [ \Omega_\psi], \cdots, \frac{d^4}{d\psi^4} [ \Omega_\psi] \) are linearly related. Thus, so are their integrals over any 3-cycle.
Idea of proof. We view $\Omega_\psi$ and its derivatives as residues. Let

$$\overline{\Omega} = \sum_{i=0}^{4} (-1)^i x_i dx_0 \wedge \cdots \wedge dx_i \wedge \cdots \wedge dx_4$$

be a form on $\mathbb{C}^5$. It is homogeneous of degree 5 (not 0), so we need to multiply by something of degree $-5$ to get a form on $\mathbb{P}^4$. If $f$, $g$ are homogeneous, $\deg f = \deg g + 5$, then $\frac{\delta \Omega}{f}$ is a meromorphic 4-form on $\mathbb{P}^4$. For instance, $\frac{5\psi \Omega}{f}$ has poles along $X_\psi$. Now, given a 4-form with poles along some hypersurface $X$, it has a residue on $X$ which is ideally a 3-form on $X$, but is at least a class in $H^3(X, \mathbb{C})$.

Recall from complex analysis, if $\phi(z)$ has a pole at 0, $\text{res}_0(\phi) = \frac{1}{2\pi i} \int_\Gamma \phi(z) dz$. Now, let’s say that we have a 3-cycle $C$ in $X$: we can associate a “tube” 4-cycle in $\mathbb{P}^4$ which is the preimage of $C$ in the boundary of a tubular neighborhood of $X$. Then

$$\int_C \text{res}_X \left( \frac{g\overline{\Omega}}{f} \right) := \frac{1}{2\pi i} \int_\Gamma \frac{g\overline{\Omega}}{f}$$

If we only have simple poles along $X$, we get a 3-form characterized by

$$\text{res}_X \left( \frac{g\overline{\Omega}}{f} \right) \wedge df = g\overline{\Omega}$$

at any point of $X$.

Now, $\Omega_\psi = \text{res}_{X_\psi} \left( \frac{5\psi \Omega}{f_\psi} \right)$, and differentiating $k$ times gives

$$\frac{\partial^k}{\partial \psi^k} [\Omega_\psi] = \text{res}_{X_\psi} \left( \frac{g_k \overline{\Omega}}{f_k+1} \right)$$

Thus we can express

$$\Theta^4[\Omega_\psi] = \text{res}_{X_\psi} \left( \frac{g_\Theta \overline{\Omega}}{f_\Theta} \right)$$

for some $g_\Theta$, and write $5z(5\Theta + 1) \cdots (5\Theta + 4)|\Omega_\psi|$ in the same form.

We compare the residues of forms with order 5 poles along $X_\psi$ using Griffiths pole order reduction. Assume that $\phi$ is a 3-form with poles of order $\ell$ along $X_\psi$,

$$\phi = \frac{1}{f_\psi} \sum_{i<j} (-1)^{i+j} (x_ig_j - x_jg_i) dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_4$$

with $\deg (g_0 \cdots g_4) = 5\ell - 4$, then

$$d\phi = \frac{1}{f_\psi^{\ell+1}} \left( \ell \sum_j g_j \frac{\partial f_\psi}{\partial x_j} - f_\psi \sum_j \frac{\partial g_j}{\partial x_j} \right) \overline{\Omega}$$
In particular, if we have something of the form \((\sum g_j \frac{\partial f}{\partial x_j})\frac{\Omega}{f^{s+1}}\) (the Jacobian ideal is the span of \(\{\frac{\partial f}{\partial x_j}\}\)), it can be written as something with a lower order pole plus something exact. We obtain our result iteratively, showing in each stage that the top order term belongs to the Jacobian ideal, and reduce to a lower order term. When we get to order 1, we find that the residue is 0. \(\Box\)

There is a theory of differential equations with regular singular points, i.e. differential equations of the form

\[
\Theta^s f + \sum_{j=0}^{s-1} B_j(z) \Theta^j f = 0
\]

where \(\Theta = z \frac{d}{dz}\) and \(B_j(z)\) are meromorphic functions which are holomorphic at \(z = 0\). As with solving ordinary differential equations, we reduce to a 1st order system of differential equations \(\Theta w(z) = A(z)w(z)\), where

\[
A(z) = \begin{pmatrix}
0 & 1 & & \\
& 0 & 1 & \\
& & \ddots & \ddots \\
&-B_0(z) & \cdots & \cdots & -B_{s-1}(z)
\end{pmatrix},
\]

\[
w(z) = \begin{pmatrix}
f(z) \\
\Theta f(z) \\
\vdots \\
\Theta^{s-1} f(z)
\end{pmatrix}
\]

The fundamental theorem of these differential equations states that there exists a constant \(s \times s\) matrix \(R\) and an \(s \times s\) matrix of holomorphic functions \(S(z)\) s.t.

\[
\Phi(z) = S(z) \exp((\log z)R) = S(z)(\text{id} + (\log z)R + \frac{\log^2 z}{2}R^2 + \cdots)
\]

is a fundamental system of solutions to \(\Theta w(z) = A(z)w(z)\), and moreover if \(A(0)\) doesn’t have distinct eigenvalues differing by an integer, we can take \(R = A(0)\). This \(\Phi\) is multivalued, and \(z \mapsto e^{2\pi i z}\) gives \(\Phi(z) \mapsto \Phi(z)e^{2\pi i R}\) (where \(e^{2\pi i R}\) is the monodromy).

In our case, \(\mathcal{D}\phi = \Theta^4 \phi - 5\Theta (5\Theta + 1) \cdots (5\Theta + 4) \phi = 0\), so the coefficient of \(\Theta^4\) is \(1 - 5^5 z\), and the coefficients of \(\Theta^0, \cdots, \Theta^3\) are constant multiples of \(z\). Then

\[
\Theta^4 \phi - \frac{5z}{1 - 5^5 z} P_3(\Theta) \cdot \phi = 0
\]

where \(P_3\) is independent of \(z\). Then

\[
R = A(0) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
is nilpotent, and our assumption holds. The corresponding monodromy is

\[
T = e^{2\pi i R} = \begin{pmatrix}
1 & 2\pi i & \frac{(2\pi i)^2}{2} & \frac{(2\pi i)^3}{6} \\
0 & 1 & 2\pi i & \frac{(2\pi i)^2}{2} \\
0 & 0 & 1 & 2\pi i \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(19)

If \( \omega(z) = \int_\beta \tilde{\Omega}_\phi \) is a period, then it is a solution of the Picard-Fuchs equation, and thus a linear combination of \( \Phi(z) \)’s. There exists a basis \( b_1, \ldots, b_4 \) of \( H_3(X, \mathbb{C}) \) s.t. \( \int_{b_i} \tilde{\Omega}_\phi = \Phi(z) \). The monodromy action in this basis is \( T \) (\( T \) maximally unipotent implies that \( 0 \) is LSCL).

1.1. More periods of \( \tilde{\Omega}_\phi \). The first fundamental solution we obtained is \( \phi_0 = \Phi(z)_{11} \), which is invariant under monodromy and regular at \( z = 0 \). Since \( \dim \ker (T - \text{id}) = 1 \), it is unique up to scaling, and \( \phi_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!z^n}{(n!)^5} \). We next obtain \( \phi_1 = \Phi(z)_{12} \) s.t. \( \phi_1(e^{2\pi i}z) = \phi_1(z) + 2\pi i \phi_0(z) \), which is unique up to multiples of \( \phi_0 \). Since \( \Phi(z) = S(z) \exp(R \log z) \), \( \phi_1(z) = \phi_0(z) \log z + \tilde{\phi}(z) \), with \( \tilde{\phi}(z) \) holomorphic. Now

\[
\Theta^j(f(z) \log z) = (\Theta^j f) \log z + j(\Theta^{j-1} f)
\]

(20)

If we write \( F(x) = x^4 - 5z \prod_{j=1}^4 (5x + j) \), then

\[
\mathcal{D}\phi_1(z) = F(\Theta)(\phi_0(z) \log z + \tilde{\phi}(z))
\]

(21)

\[
= (F(\Theta)\phi_0) \log z + F'(\Theta)\phi_0 + F'(\Theta)\tilde{\phi}
\]

Since \( 0 = \mathcal{D}\phi_0 = \mathcal{D}\phi_1 \), we find \( \mathcal{D}\tilde{\phi}(z) = -F'(\Theta)\phi_0(z) \). This gives a recurrence relation on the coefficients of \( \tilde{\phi}(z) \), and one obtains:

\[
\tilde{\phi}(z) = 5 \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} \left( \sum_{j=n+1}^{5n} \frac{1}{j} \right) z^n
\]

(22)

We want canonical coordinates on the moduli space of complex structures: there are \( \beta_0, \beta_1 \in H_3(X, \mathbb{Z}) \), with monodromy \( \beta_0 \mapsto \beta_0, \beta_1 \mapsto \beta_1 + \beta_0 \), and

\[
\int_{\beta_0} \tilde{\Omega} = C\phi_0(z)
\]

(23)

\[
\int_{\beta_1} \tilde{\Omega} = C'\phi_0(z) + C''\phi_1(z)
\]
The monodromy acts on the latter by $\int_{\beta_1} \check{\Omega} \mapsto \int_{\beta_1 + \beta_0} \check{\Omega}$, implying that $2\pi i C'' = C$. Thus, the canonical coordinates are

$$w = \frac{\int_{\beta_1} \check{\Omega}}{\int_{\beta_0} \check{\Omega}} = \frac{C'}{C} + \frac{1}{2 \pi i} \phi_1 + \frac{1}{2 \pi i} \phi_0$$

(24)

$$q = \exp(2\pi i w) = c_2 z \exp \left( \frac{\check{\phi}(z)}{\phi_0(z)} \right)$$