One horse has 20% chance to win another has 80% chance

$10000 is put on the first one and $50000 on the second

If odds are set 4-1:
- Bookie may gain $10000 (if first horse wins)
- Bookie may loose $2500 (if second horse wins)
- Bookie expects to make 0.2 * (10000) + 0.8 * (-2500) = 0

If odds are set 5-1:
- Bookie will not lose or gain money no matter which horse wins
We are interested in finding prices of various derivatives.

**Forward contract** pays $S-K$ at time $T$:

\[ S(t) = 80, \quad K = 88.41, \quad T = 2 \text{ (years)} \]
European Call option pays $\max(S-K, 0)$ at time $T$

\[ S(t) = 80, \; K = 80, \; T = 2 \text{ (years)} \]
European Put option pays $\max(K-S, 0)$ at time $T$

$S(t)=80$, $K=80$, $T=2$ (years)
Risk Neutral Valuation: Introduction

- Given current price of the stock and assumptions on the dynamics of stock price, there is no uncertainty about the price of a derivative

- The price is defined only by the price of the stock and not by the risk preferences of the market participants

- Mathematical apparatus allows to compute current price of a derivative and its risks, given certain assumptions about the market
Consider *Forward* contract which pays $S-K$ in time $dt$. One could think that its strike $K$ should be defined by the “real world” transition probability $p$:

$$p(S_1-K)+(1-p)(S_2-K)=pS_1+(1-p)S_2-K$$

$$K_0 = pS_1+(1-p)S_2$$

If $p=1/2$, $K_0=(S_1+S_2)/2$
Risk Neutral Valuation: Replicating Portfolio

Consider the following strategy:

1. Borrow $S_0$ to buy the stock. Enter Forward contract with strike $K_0$

2. In time $dt$ deliver stock in exchange for $K_0$ and repay $S_0e^{rdt}$

- If $K_0 > S_0e^{rdt}$ we made riskless profit
- If $K_0 < S_0e^{rdt}$ we definitely lost money \[ \Rightarrow K_0 = S_0e^{rdt} \]

Current price of a derivative claim is determined by current price of a portfolio which exactly replicates the payoff of the derivative at the maturity

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Suppose our economy includes stock $S$, riskless money market account $B$ with interest rate $r$ and derivative claim $f$. Assume that only two outcomes are possible in time $dt$:

$$
\begin{align*}
S_0, B_0, f_0 & \quad \text{with probability } 1-p \\
S_1, B_0e^{rdt}, f_1 & \quad \text{with probability } p \\
S_2, B_0e^{rdt}, f_2 & \quad \text{with probability } 1-p
\end{align*}
$$

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Risk Neutral Valuation: One step binomial tree

For a general derivative claim $f$, find $a$ and $b$ such that

$$f_1 = aS_1 + bB_0 e^{rdt}$$
$$f_2 = aS_2 + bB_0 e^{rdt}$$

Then

$$f_0 = as_0 + bB_0$$

Easy to see that

$$a = \frac{f_1 - f_2}{S_1 - S_2}, \quad b = \frac{S_1 f_2 - S_2 f_1}{(S_1 - S_2)B_0 e^{rdt}}$$

$$f_0 = e^{-rdt} \left( S_0 e^{rdt} \frac{f_1 - f_2}{S_1 - S_2} + \frac{S_1 f_2 - S_2 f_1}{S_1 - S_2} \right)$$

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One should notice that

\[ f_0 = e^{-rdt} \left( f_1 \frac{S_0 e^{rdt} - S_2}{S_1 - S_2} + f_2 \frac{S_1 - S_0 e^{rdt}}{S_1 - S_2} \right) \]

\[ f_0 = e^{-rdt} (f_1 q + f_2 (1 - q)) \]

where

\[ q = \frac{(S_0 e^{rdt} - S_2)}{(S_1 - S_2)}, \quad 0 < q < 1 \]

Moreover

\[ S_1 q + S_2 (1-q) = e^{rdt} S_0 \]
Risk Neutral Valuation: Continuous case

\[ f_t = e^{-r(T-t)} E_Q[f_T] \]

\( Q \) is the risk neutral (martingale) measure under which

\[ S_0 = e^{-rt} E_Q[S_T] \]
Assume that the stock has log-normal dynamics:

\[
dS = \mu S dt + \sigma S dW
\]

Where \( dW \) is normally distributed with mean 0 and standard deviation \( \sqrt{dt} \) (i.e. \( W \) is a Brownian Motion)

We want to find a replicating portfolio such that

\[
df = a dS + b dB
\]

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Black-Scholes equation

Use Ito’s formula:

\[
df(S, t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} (dS)^2
\]

\[(dS)^2 = \sigma^2 S^2 dt\]

(analogous to first order Taylor expansion, up to \(dt\) term)
Black-Scholes equation

\[ df = adS + bdB \]

Substitute \( dS, df, dB = rBdt \) and \((dS)^2\)

\[
\left( \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} \mu S + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma SdW = (a \mu S + brB)dt + a \sigma SdW
\]

Compare terms

\[
a = \frac{\partial f}{\partial S}, \quad brB = \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2
\]

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Black-Scholes equation

$bB=f-aS$ is deterministic and as $dB=rBdt$

\[ d(f-aS)=r(f-aS)dt \]

Substituting once again \( df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 dt \) and \( a = \frac{\partial f}{\partial S} \)

we obtain the Black-Scholes equation

\[ \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 + \frac{\partial f}{\partial S} rS - rf = 0 \]

Fisher Black, Myron Scholes – paper 1973

Myron Scholes, Robert Merton – Nobel Prize 1997

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Black-Scholes equation

- Any tradable derivative satisfies the equation
- There is no dependence on actual drift $\mu$
- We have a hedging strategy (replicating portfolio)
- By a change of variables Black-Scholes equation transforms into heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

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Boundary and final conditions are determined by the pay-off of a specific derivative

For European Call

\[ C(S,T) = \max(S-K,0) \]

\[ C(0,t) = 0, C(\infty,t) \approx S \]

For European Put

\[ P(S,T) = \max(K-S,0) \]

\[ P(0,t) = Ke^{-r(T-t)}, P(\infty,t) = 0 \]
Black-Scholes equation

For European Call/Put the equation can be solved analytically

\[ C_t = e^{-r(T-t)} \left( e^{r(T-t)} SN(d_1) - KN(d_2) \right) \]
\[ P_t = e^{-r(T-t)} \left( KN(-d_2) - e^{r(T-t)} SN(-d_1) \right) \]

where

\[ d_1 = \frac{\ln(S/K) + (r + \sigma^2 / 2)(T-t)}{\sigma \sqrt{T-t}} \]
\[ d_2 = \frac{\ln(S/K) + (r - \sigma^2 / 2)(T-t)}{\sigma \sqrt{T-t}} \]
\[ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du \]

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Black-Scholes: Risk Neutral Valuation

$$f_t = e^{-r(T-t)}E_Q[f_T]$$

$Q$ is the risk neutral measure under which

$$dS = rSdt + \sigma SdW$$

$$PDF(S_T) = \frac{1}{\sigma S \sqrt{2\pi T}} \exp \left( - \frac{\ln(S_T / S_t) - \left( r - \frac{\sigma^2}{2} \right)(T-t)}{2\sigma^2(T-t)} \right)$$

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Black-Scholes equation

For more complicated options or more general assumptions numerical methods have to be used:

- Finite difference methods
- Tree methods (equivalent to explicit scheme)
- Monte Carlo simulations
Modern financial services business makes use of

- PDE
- Numerical methods
- Stochastic Calculus
- Simulations
- Statistics
- Much, much more
## Risk Neutral Valuation: Example

![Option Monitor: IBM Business Machines Corp](image)

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<th>Ticker</th>
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Source: Bloomberg L.P.
Digital option pays 1 if $S > K$ at time $T$

$S(t) = 80$, $K = 80$, $T = 2$ (years)
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