18.S66 PROBLEMS #3
Spring 2003

Beginning with this assignment we will (subjectively) indicate the difficulty level of each problem as follows:

1. easy
2. moderately difficult
3. difficult.

In general, these difficulty ratings are based on the assumption that the solutions to the previous problems are known.

A partition $\lambda$ of $n \geq 0$ (denoted $\lambda \vdash n$ or $|\lambda| = n$) is an integer sequence $(\lambda_1, \lambda_2, \ldots)$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $\sum \lambda_i = n$. Trailing 0’s are often ignored, e.g., $(4, 3, 3, 1, 1)$ represents the same partition of 12 as $(4, 3, 3, 1, 1, 0, 0, 0, \ldots)$ or $(4, 3, 3, 1, 1, 0, 0, \ldots)$. The terms $\lambda_i > 0$ are called the parts of $\lambda$. The conjugate partition to $\lambda$, denoted $\lambda'$, has $\lambda_i - \lambda_{i+1}$ parts equal to $i$ for all $i \geq 1$. The (Young) diagram of $\lambda$ is a left-justified array of squares with $\lambda_i$ squares in the $i$th row. Notation such as $u = (2, 3) \in \lambda$ means that $u$ is the square of the diagram of $\lambda$ in the second row and third column.

69. [1] Let $\lambda$ be a partition. Then

$$\sum \limits_i (i - 1)\lambda_i = \sum \limits_i \left(\frac{\lambda_i^2}{2}\right).$$

70. [1] Let $\lambda$ be a partition. Then

$$\sum \limits_i \left[\frac{\lambda_{2i-1}}{2}\right] = \sum \limits_i \left[\frac{\lambda_{2i-1}}{2}\right]$$

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71. [1] The number of partitions of $n$ with largest part $k$ equals the number of partitions of $n$ with exactly $k$ parts.

72. [2] Fix $k \geq 1$. Let $\lambda$ be a partition. Define $f_k(\lambda)$ to be the number of parts of $\lambda$ equal to $k$, e.g., $f_3(8, 5, 5, 3, 3, 3, 2, 1, 1) = 4$. Define $g_k(\lambda)$ to be the number of integers $i$ for which $\lambda$ has at least $k$ parts equal to $i$, e.g., $g_3(8, 8, 8, 6, 6, 3, 2, 2, 1) = 2$. Then

$$\sum_{\lambda \vdash n} f_k(\lambda) = \sum_{\lambda \vdash n} g_k(\lambda).$$

73. [2] The number of partitions of $n$ with odd parts equals the number of partitions of $n$ with distinct parts.

74. [2] Let $\sigma(n)$ denote the sum of all (positive) divisors of $n \in \mathbb{P}$; e.g., $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. Let $p(n)$ denote the number of partitions of $n$ (with $p(0) = 1$). Then

$$n \cdot p(n) = \sum_{i=1}^{n} \sigma(i) p(n - i).$$

75. [2] The number of self-conjugate partitions of $n$ equals the number of partition of $n$ into distinct odd parts.

76. [3] Let $f(n)$ be the number of partitions of $n$ into an even number of parts, all distinct. Let $g(n)$ be the number of partitions of $n$ into an odd number of parts, all distinct. For instance, $f(7) = 3$, corresponding to $6 + 1 = 5 + 2 = 4 + 3$, and $g(7) = 2$, corresponding to $7 = 4 + 2 + 1$. Then

$$f(n) - g(n) = \begin{cases} (-1)^k, & \text{if } n = k(3k \pm 1)/2 \text{ for some } k \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

**NOTE.** This result is usually stated in generating function form, viz.,

$$\prod_{n \geq 1} (1 - x^n) = 1 + \sum_{k \geq 1} (-1)^k \left( x^{k(3k-1)/2} + x^{k(3k+1)/2} \right),$$

and is known as *Euler’s pentagonal number formula.*
77. [2] Let \( f(n) \) (respectively, \( g(n) \)) be the number of partitions \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of \( n \) into distinct parts, such that the largest part \( \lambda_1 \) is even (respectively, odd). Then

\[
f(n) - g(n) = \begin{cases} 
1, & \text{if } n = k(3k + 1)/2 \text{ for some } k \geq 0 \\
-1, & \text{if } n = k(3k - 1)/2 \text{ for some } k \geq 1 \\
0, & \text{otherwise.}
\end{cases}
\]

78. [3] For \( n \in \mathbb{N} \) let \( f(n) \) (respectively, \( g(n) \)) denote the number of partitions of \( n \) into distinct parts such that the smallest part is odd and with an even number (respectively, odd number) of even parts. Then

\[
f(n) - g(n) = \begin{cases} 
1, & \text{if } n \text{ is a square} \\
0, & \text{otherwise.}
\end{cases}
\]

79. (a) (*) The number of partitions of \( n \) into parts \( \equiv \pm 1 \pmod{5} \) is equal to the number of partitions of \( n \) whose parts differ by at least 2.

(b) (*) The number of partitions of \( n \) into parts \( \equiv \pm 2 \pmod{5} \) is equal to the number of partitions of \( n \) whose parts differ by at least 2 and for which 1 is not a part.

**Note.** This is the combinatorial formulation of the famous Rogers-Ramanujan identities. One of the known proofs of this result has been converted into a complicated recursive bijection. What is wanted is a “direct” bijection whose inverse is easy to describe.

80. [3] The number of partitions of \( n \) into parts \( \equiv 1, 5, \text{ or } 6 \pmod{8} \) is equal to the number of partitions into parts that differ by at least 2, and such that odd parts differ by at least 4.

81. [3] A lecture hall partition of length \( k \) is a partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \) (some of whose parts may be 0) satisfying

\[
0 \leq \frac{\lambda_k}{1} \leq \frac{\lambda_{k-1}}{2} \leq \cdots \leq \frac{\lambda_1}{k}.
\]

The number of lecture hall partitions of \( n \) of length \( k \) is equal to the number of partitions of \( n \) whose parts come from the set \( \{1, 3, 5, \ldots, 2k-1\} \) (with repetitions allowed).
82. (*) The Lucas numbers $L_n$ are defined by $L_1 = 1$, $L_2 = 3$, $L_{n+1} = L_n + L_{n-1}$ for $n \geq 2$. Let $f(n)$ be the number of partitions of $n$ all of whose parts are Lucas numbers $L_{2n+1}$ of odd index. For instance, $f(12) = 5$, corresponding to

$$
1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 \\
4 + 1 + 1 + 1 + 1 + 1 + 1 + 1 \\
4 + 4 + 1 + 1 + 1 \\
4 + 4 + 4 \\
11 + 1
$$

Let $g(n)$ be the number of partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $\lambda_i/\lambda_{i+1} > \frac{1}{2}(3 + \sqrt{5})$ whenever $\lambda_{i+1} > 0$. For instance, $g(12) = 5$, corresponding to

$$
12, \ 11 + 1, \ 10 + 2, \ 9 + 3, \ 8 + 3 + 1.
$$

Then $f(n) = g(n)$ for all $n \geq 1$.

83. [2.5] Let $A(n)$ denote the number of partitions $(\lambda_1, \ldots, \lambda_k) \vdash n$ such that $\lambda_k > 0$ and

$$
\lambda_i > \lambda_{i+1} + \lambda_{i+2}, \ 1 \leq i \leq k - 1
$$

(with $\lambda_{k+1} = 0$). Let $B(n)$ denote the number of partitions $(\mu_1, \ldots, \mu_j) \vdash n$ such that

- Each $\mu_i$ is in the sequence $1, 2, 4, \ldots, g_m, \ldots$ defined by
  $$
g_1 = 1, \ g_2 = 2, \ g_m = g_{m-1} + g_{m-2} + 1 \text{ for } m \geq 3.
$$
- If $\mu_1 = g_m$, then every element in $\{1, 2, 4, \ldots, g_m\}$ appears at least once as a $\mu_i$.

Then $A(n) = B(n)$ for all $n \geq 1$.

*Example.* $A(7) = 5$ because the relevant partitions are $(7)$, $(6, 1)$, $(5, 2)$, $(4, 3)$, $(4, 2, 1)$, and $B(7) = 5$ because the relevant partitions are $(4, 2, 1)$, $(2, 2, 2, 1)$, $(2, 2, 1, 1, 1)$, $(2, 1, 1, 1, 1, 1)$, $(1, 1, 1, 1, 1, 1, 1)$.

84. (*) Let $S \subseteq \mathbb{P}$ and let $p(S, n)$ denote the number of partitions of $n$ whose parts belong to $S$. Let

$$
S = \pm\{1, 4, 5, 6, 7, 9, 10, 11, 13, 15, 16, 19 \text{ (mod 40)}\} \\
T = \pm\{1, 3, 4, 5, 9, 10, 11, 14, 15, 16, 17, 19 \text{ (mod 40)}\},
$$
where

$$\pm\{a, b, \ldots \pmod{m}\} = \{n \in \mathbb{P} : n \equiv \pm a, \pm b, \ldots \pmod{m}\}.$$  

Then \(p(S, n) = p(T, n - 1)\) for all \(n \geq 1\).

**Note.** In principle the known proof of this result and of Problem 85 below can be converted into a complicated recursive bijection, as was done for Problem 79. Just as for Problem 79, what is wanted is a “direct” bijection whose inverse is easy to describe. To my knowledge no one has tried to give a bijective solution to this problem and the next, so perhaps they are not so difficult.

85. (*) Let

\[
S = \pm\{1, 4, 5, 6, 7, 9, 11, 13, 16, 21, 23, 28 \pmod{66}\}
\]

\[
T = \pm\{1, 4, 5, 6, 7, 9, 11, 14, 16, 17, 27, 29 \pmod{66}\}.
\]

Then \(p(S, n) = p(T, n)\) for all \(n \geq 1\) except \(n = 13\) (!).

86. [1.5] Prove the following identities by interpreting the coefficients in terms of partitions.

\[
\prod_{i \geq 1} \frac{1}{1 - qx^i} = \sum_{k \geq 0} \frac{x^k q^k}{(1 - x)(1 - x^2) \cdots (1 - x^k)}
\]

\[
\prod_{i \geq 1} \frac{1}{1 - qx^i} = \sum_{k \geq 0} \frac{x^k q^k}{(1 - x) \cdots (1 - x^k)(1 - qx) \cdots (1 - qx^k)}
\]

\[
\prod_{i \geq 1} (1 + qx^i) = \sum_{k \geq 0} \frac{x^{k+1}}{\binom{k+1}{2}} q^k
\]

\[
\prod_{i \geq 1} (1 + qx^{2i-1}) = \sum_{k \geq 0} \frac{x^k q^k}{(1 - x^2)(1 - x^4) \cdots (1 - x^{2k})}.
\]

87. [3] Show that

\[
\sum_{n=-\infty}^{\infty} x^n q^{n^2} = \prod_{k \geq 1} (1 - q^{2k})(1 + xq^{2k-1})(1 + x^{-1}q^{2k-1}).
\]

This famous result is *Jacobi’s triple product identity.*
88. [3] Let \( f(n) \) be the number of partitions of \( 2n \) whose Ferrers diagram can be covered by \( n \) edges, each connecting two adjacent dots. For instance, \((4, 3, 3, 3, 1)\) can be covered as follows:

\[
\begin{array}{cccc}
\hline
& & & \\
\hline
& & & \\
& & & \\
& & & \\
& & & \\
\hline
\end{array}
\]

Then \( f(n) \) is equal to the number of ordered pairs \((\lambda, \mu)\) of partitions satisfying \(|\lambda| + |\mu| = n\).

89. (*) Given a partition \( \lambda \) and \( u \in \lambda \), let \( a(u) \) (called the arm length of \( u \)) denote the number of squares directly to the right of \( u \) (in the diagram of \( \lambda \)), counting \( \lambda \) itself exactly once. Similarly let \( l(u) \) (called the leg length of \( u \)) denote the number of squares directly below \( u \), counting \( u \) itself once. Thus if \( u = (i, j) \) then \( a(u) = \lambda_i - j + 1 \) and \( l(u) = \lambda'_j - i + 1 \). Define

\[
\gamma(\lambda) = \# \{ u \in \lambda : a(u) - l(u) = 0 \text{ or } 1 \}.
\]

Then

\[
\sum_{\lambda \vdash n} q^{\gamma(\lambda)} = \sum_{\lambda \vdash n} q^{\ell(\lambda)},
\]

where \( \ell(\lambda) \) denotes the length (number of parts) of \( \lambda \).

90. [2.5] If \( 0 \leq k < \lfloor n/2 \rfloor \), then \( \binom{n}{k} \leq \binom{n}{k+1} \).

**NOTE.** To prove an inequality \( a \leq b \) combinatorially, find sets \( A, B \) with \( \#A = a \), \( \#B = b \), and either an injection (one-to-one map) \( f : A \to B \) or a surjection (onto map) \( g : B \to A \).

91. [2.5] Let \( 1 \leq k \leq n - 1 \). Then \( \binom{n}{k}^2 \geq \binom{n}{k-1} \binom{n}{k+1} \). Note that this result is even stronger than Problem 90 above (assuming \( \binom{n}{k} = \binom{n}{n-k} \)) [why?].

92. [1] Let \( p(j, k, n) \) denote the number of partitions of \( n \) with at most \( j \) parts and with largest part at most \( k \). Then \( p(j, k, n) = p(j, k, jk - n) \).
\[ \sum_{n=0}^{jk} p(j, k, n)q^n = \binom{j+k}{j}, \]

where \( \binom{m}{i} \) denotes the \emph{q-binomial coefficient}:

\[ \binom{m}{i} = \frac{(1 - q^m)(1 - q^{m-1}) \cdots (1 - q^{m-i+1})}{(1 - q^i)(1 - q^{i-1}) \cdots (1 - q)}. \]

93. [3] Continuing the previous problem, if \( n < \frac{jk}{2} \) then \( p(j, k, n) \leq p(j, k, n+1) \).

\textbf{NOTE.} A (difficult) combinatorial proof is known. What is really wanted, however, is an injection \( f : A_n \to A_{n+1} \), where \( A_m \) is the set of partitions counted by \( p(j, k, m) \), such that for all \( \lambda \in A_n \), \( f(\lambda) \) is obtained from \( \lambda \) by adding 1 to a single part of \( \lambda \). It is known that such an injection \( f \) exists, but no explicit description of \( f \) is known.

94. [1] Let \( \bar{p}(k, n) \) denote the number of partitions of \( n \) into distinct parts, with largest part at most \( k \). Then

\[ \bar{p}(k, n) = \bar{p}(k, \left( \frac{k+1}{2} \right) - n). \]

\textbf{NOTE.} It is easy to see that

\[ \sum_{n=0}^{\left( \frac{k+1}{2} \right)} \bar{p}(k, n)q^n = (1 + q)(1 + q^2) \cdots (1 + q^k). \]

95. (*) Continuing the previous problem, if \( n < \frac{1}{2} \left( \frac{k+1}{2} \right) \) then \( \bar{p}(k, n) \leq \bar{p}(k, n+1) \).

\textbf{NOTE.} As in Problem 93 it would be best to give an injection \( g : B_n \to B_{n+1} \), where \( B_m \) is the set of partitions counted by \( \bar{p}(k, m) \), such that for all \( \lambda \in B_n \), \( f(\lambda) \) is obtained from \( \lambda \) by adding 1 to a single part of \( \lambda \). It is known that such an injection \( g \) exists, but no explicit description of \( g \) is known. However, unlike Problem 93, no explicit injection \( g : B_n \to B_{n+1} \) is known.
96. [2] A partition $\pi$ of a set $S$ is a collection of nonempty pairwise disjoint subsets (called the blocks of $\pi$) of $S$ whose union is $S$. Let $B(n)$ denote the number of partitions of an $n$-element set. $B(n)$ is called a Bell number. For instance, $B(3) = 5$, corresponding to the partitions (written in an obvious shorthand notation) 1-2-3, 12-3, 13-2, 1-23, 123. The number of partitions of $[n]$ for which no block contains two consecutive integers is $B(n - 1)$. 