2.004 Dynamics and Control II
Spring 2008

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Reading:
  - Class Handout: *Modeling Part 1: Energy and Power Flow in Linear Systems*
    Sec. 1 (Introduction)
    Sec. 4 (Electrical System Elements)

1 Modeling of Physical Systems

We will develop a unified modeling method that will allow the generation of a transfer function in several energy domains. The method is based on an extension of impedance concepts used in electrical systems. For that reason we will start with electrical systems.

*Lumped parameter modeling* involves approximating physical elements (where the parameters may be distributed in space) with discrete elements that are assumed to be concentrated at points (or “lumped”) in space. For example, a real electrical inductor is made from a coil of wire.

![Image of a coil with text: The coil has properties of inductance (energy storage in the magnetic field) and resistance (energy dissipation).]

The coil has the primary property of electrical *inductance*, BUT it also has property of *resistance* associated with the wire - this resistance is distributed throughout the coil. In lumped parameter modeling we approximate the coil as having two elements - an inductance L and a lumped resistance R.

2 Modeling Electrical Systems

**Step 1: Identify the variables to be used.** We will concentrate on the use of *power variables*, that is a pair of variables whose product is power. In electrical systems:

\[ P = vi \]

where \( P \) is power, \( v \) is the voltage drop (volts) across an element, and \( i \) is the current (amps) through an element. We therefore use \( v \) and \( i \) as our modeling variables.

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Step 2: Identify the "primitive" lumped modeling elements. In each energy domain we will identify 3 elements, two elements that store energy, and an element that dissipates energy. In the electrical domain the energy storage elements are the capacitor and the inductor, and the dissipative element is the resistor.

The Capacitor: The capacitor stores energy in the electrostatic field between its "plates".

For a capacitor the current and voltage drop are related by

\[ i = C \frac{dv}{dt} \]

where \( C \) is defined as the capacitance (Farad), or alternatively

\[ v = \frac{1}{C} \int_0^t i dt + v(0) \]

Also, if we define charge as \( q = \int_0^t i dt \) then

\[ v = \frac{1}{C} q \]

For a parallel plate capacitor, where the plates have area \( A \), and are separated by a distance \( d \)

\[ C \propto \frac{A}{d} \implies C = \frac{\epsilon A}{d} \]

where \( \epsilon \) is defined to be the permittivity of the (non-conducting) dielectric material between the plates.

Note that power can flow in and out of a capacitor, since

\[ P = vi = C v \frac{dv}{dt} \]

and \( v \) and \( dv/dt \) may have same or opposite signs. The capacitor is therefore an energy storage element. The energy stored in a capacitor is

\[ E_C(t) = \int_0^t P dt + E(0) = \frac{1}{2} C v^2. \]
Notes:
1) The Farad is a very large unit. Most practical capacitors are measured in units of
   • microfarads ($\mu$F) = $10^{-6}$F,
   • nanofarads (nF) = $10^{-9}$F, or
   • picofarads (pF) = $10^{-12}$F
2) A capacitor will not pass a dc current. Current only flows when the applied voltage is changing.

The Inductor:

Inductance is a property that represents energy stored in the electromagnetic field in a current carrying conductor. A practical inductor consists of a coil of wire

\[ L = \frac{\mu N^2 A}{l} \]

Helically wound coil

Lenz’ Law relates the voltage across an inductor to the current flowing through it:

\[ v = L \frac{di}{dt} \]

or

\[ i = \frac{1}{L} \int_0^t v dt + i(0) \]

where \( L \) is the inductance, with units of the Henry. Power can flow in and out of an inductor since

\[ P = vi = Li \frac{di}{dt} \]

and \( i \) and \( di/dt \) may have the same or opposite signs. The stored energy is

\[ E = \int_0^t P dt + E(0) = \frac{1}{2} Li^2 \]
Notes:
1) The Henry is a very large unit. Practical inductors have values in
   • millihenrys (mH) - $10^{-3}$H
   • microhenrys ($\mu$H) - $10^{-6}$H
2) Notice that a pure inductance will have $v = 0$ if $\frac{di}{dt} = 0$ (a dc current)
3) Practical inductors are wound with (copper) wire and will have a finite resistance $R$. The lumped model including both phenomena is

The relative importance of the resistance is dependent on the operating conditions.

The Resistor:

![Resistor Diagram]

The current and voltage in a resistor are governed by Ohm’s law:

$$V = iR \quad \text{or} \quad i = \frac{1}{R}v$$

Resistance is a dissipative property since $v$ and $i$ always have the same sign, and the power is always positive

$$P = i^2R = \frac{v^2}{R} > 0$$

Energy always flows into a resistor, and cannot be recovered.

Summary: In modeling electrical systems we use three passive modeling elements:

![Resistor Diagram]

The capacitor and inductor are energy storage elements in that power can flow into the element, and recovered. The resistor is a dissipative element because the voltage an current are algebraically related and power always flows into the element.

Sources: In addition to the three passive elements we will use two ideal electrical power sources:
The Voltage Source:

\[ v_2(t) - v_1(t) = V(t) \]

A voltage source will maintain the voltage at its terminals regardless of the current it must supply to the connected load.

The Current Source:

\[ I_s(t) = I_0 \]

A current source will maintain the current supplied to the attached circuit regardless of the voltage at it must generate to do so.