Reading:

- Nise: Sec. 2.4 (pages 45–55)
- Class Handout: *Modeling Part 1: Energy and Power Flow in Linear Systems*
  Sec. 1 (Introduction)
  Sec. 4 (Electrical System Elements)

1 Modeling Electrical Systems (continued)

In Lecture 5 we examined the primitive electrical elements (capacitors inductors and resistors), and sources (voltage source and current source). We now look at how these elements behave when connected together in a circuit.

Interconnection Laws:

(a) Kirchhoff’s Current Law (KCL): The sum of currents flowing into(or out of) a junction is zero. In the figure below, at the circled junction we sum the currents into the junction to find

\[ i_1 - i_2 - i_3 = 0 \]

We will define a junction as a *node*, and if there are \( n \) circuit branches attached to a node

\[ \sum_{i=1}^{n} i_n = 0 \]

where we define the convention that positive current flow is into the node.

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Kirchoff’s Voltage Law (KVL): The sum of voltage drops around any closed loop in a circuit is zero. The assumed sign convention for the voltage drop on each element must be defined. Two clockwise loops are shown in the figure below. For loop (1)

\[ v_R + v_C - V_s(t), \]

while for loop (2)

\[ v_L - v_C(t) = 0. \]

Electrical Impedance:

Define the \textit{impedance} of an element or passive circuit as a \textit{transfer function} relating current \(I(s)\) to voltage \(V(s)\) at its terminals:

\[ Z(s) = \frac{V(s)}{I(s)} \]

In addition we can define the \textit{admittance} \(Y(s)\) as the reciprocal of the impedance:

\[ Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)} \]

The Impedance of Passive Electrical Elements

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(a) The Capacitor:  

For the capacitor  
\[ i = C \frac{dv}{dt} \]

Taking the Laplace Transform:  
\[ I(s) = CsV(s) \]

or the admittance \( Y_C(s) = sC \).

\[ Z_C(s) = \frac{V(s)}{I(s)} = \frac{1}{sC} \]

(b) The Inductor:  

For the inductor  
\[ v = L \frac{di}{dt} \]

Taking the Laplace Transform:  
\[ V(s) = LsI(s) \]

or the admittance \( Y_L(s) = 1/sL \).

\[ Z_L(s) = \frac{V(s)}{I(s)} = sL \]

(c) The Resistor:  

For the resistor  
\[ v = Ri \]

Taking the Laplace transform  
\[ V(s) = RI(s) \]

or the admittance \( Y_R = 1/R \).

\[ Z_R(s) = \frac{V(s)}{I(s)} = R \]

**Impedance Nomenclature:** We now introduce a graphical representation that will be used to denote systems in many energy domains.
The impedance is drawn as a graph *branch* between two *nodes*. Nodes represent junctions between the elements in the circuit. The arrow on the branch indicates both the assumed direction of voltage drop across the element, and the assumed current direction.

### Example 1

The electrical circuit, consisting of a capacitor $C$, an inductor $L$, and a resistor $R$ is shown below:

The impedance graph is shown on the right. The nodes on the graph represent *points of distinct voltage* in the circuit.

### Impedance Connection Rules

(a) **Series connection:** Two or more elements are defined to be connected in series *if they share a common current*. For the two elements $Z_1$ and $Z_2$ in series below:

Using KCL at the junction between $Z_1$ and $Z_2$:

$$i_{Z_1} = i_{Z_2} = i$$

Using KVL around the loop: $v_{Z_1} + v_{Z_2} - V_s = 0$

$$V_s = iZ_1 + iZ_2$$

$$\frac{V(s)}{I(s)} = Z_{eq} = Z_1 + Z_2$$

In general with $n$ impedances $Z_i$ ($i = 1, \ldots, n$) in series:

$$Z_{eq} = \sum_{i=1}^{n} Z_i$$

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**Example 2**

For the tree elements in series below:

\[ Z_{eq} = Z_C + Z_L + Z_R = \frac{1}{sC} + sL + R \]

or expressing the impedance as a transfer function (a ratio of polynomials):

\[ Z_{eq} = \frac{V(s)}{I(s)} = \frac{LCs^2 + RCs + 1}{Cs} \]

(b) **Parallel connection:** Two or more elements are defined to be connected in parallel if they *share a common voltage*. For the two elements \( Z_1 \) and \( Z_2 \) in parallel below:

Using KVL:

\[ v_{Z_1} = v_{Z_2} = V_s \]

Using KCL at the node:

\[ i_s = i_{Z_1} + i_{Z_2} \]

\[ \frac{1}{Z_{eq}} = \frac{I}{V} = \frac{i_{Z_1} + i_{Z_2}}{V} \]

\[ \frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} \]

In general for \( n \) impedances \( Z_i \ (i = 1, \ldots, n) \) in parallel, the equivalent impedance is:

\[ \frac{1}{Z_{eq}} = \sum_{i=1}^{n} \frac{1}{Z_i} \]

Alternatively, using admittances \( Y = 1/Z \)

\[ y_{eq} = \frac{1}{Z_{eq}} = \sum_{i=1}^{n} Y_i. \]

**Note:** For \( N = 2 \) we can write

\[ \frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} = \frac{Z_1 + Z_2}{Z_1Z_2} \]

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which leads to the very common representation

\[
Z_{eq} = \frac{Z_1 Z_2}{Z_1 + Z_2}
\]

### Example 3

Find the impedance of a capacitor \(C\), and inductor \(L\) and a resistor \(R\) connected in parallel:

\[
\frac{1}{Z} = \frac{1}{1/sC} + \frac{1}{sL} + \frac{1}{R}
= sC + \frac{1}{sL} + \frac{1}{R}
= \frac{LCRs^2 + Ls + R}{RLs}
\]

\[
Z = \frac{V(s)}{I(s)} = \frac{RLs}{LCRs^2 + Ls + R}
\]

### Example 4

Find the impedance of the following circuit, assuming we should include resistance and inductance of the coil:

\[
Z = Z_1 + Z_4 \parallel (Z_2 + Z_3)
= Z_1 + \frac{Z_4(Z_2 + Z_3)}{Z_4 + Z_2 + Z_3}
= R + \frac{(1/sC)(R_L + Ls)}{1/sC + R_L + Ls}
= R + \frac{R_L + Ls}{LCs^2 + R_LCs + 1}
\]
The Voltage Divider: Consider two impedances in series with voltage \( V \) across them:

\[
Z = \frac{V(s)}{I(s)} = \frac{RLC s^2 + (RR_L C + L)s + (R + R_L)}{L C s^2 + R_L C s + 1}
\]

\[
I(s) = \frac{V(s)}{Z_1 + Z_2}
\]

and

\[
V_{Z_2}(s) = I(s)Z_2 = \frac{Z_2}{Z_1 + Z_2}V(s).
\]

Similarly

\[
V_{Z_1}(s) = \frac{Z_1}{Z_1 + Z_2}V(s).
\]

The voltage divider relationship may be used to find the transfer function of many simple systems.

**Example 5**

Find the transfer function relating \( V_0 \) to \( V_s \) in the following circuit:

\[
V_0 = \frac{Z_2}{Z_1 + Z_2}V_s = \frac{1/sC}{R + 1/sC}V_s
\]

\[
H(s) = \frac{V_0(s)}{V(s)} = \frac{1}{RCs + 1}
\]

**Example 6**

Find the transfer function relating \( V_0 \) to \( V_s \) in the following circuit:

\[6-7\]
Reduce the impedance graph to a series connection of two elements

\[ V_0 = \frac{Z_5}{Z_1 + Z_5} V_s \]

where

\[ Z_5 = Z_4 \parallel (Z_2 + Z_3) = \frac{Z_4(Z_2 + Z_3)}{Z_4 + Z_2 + Z_3} = \frac{(1/sC)(R_L + L_s)}{1/sC + R_L + Ls} \]

Using the voltage divider relationship, the transfer function is

\[ H(s) = \frac{V_0(s)}{V_s(s)} = \frac{Z_5}{Z_1 + Z_5} = \frac{R_{1L} + Ls}{R_1 + \frac{R_{1L} + Ls}{LCs^2 + R_1Cs + 1}} \]

\[ H(s) = \frac{R_L + Ls}{R_1LCs^2 + (R_1R_LC + L)s + (R_1 + R_L)} \]

**The Current Divider:** Consider two impedances in parallel:
Using KCL at the top node (a),

\[ I - i_1 - i_2 = 0 \quad \text{or} \quad i_1 + i_1 = I \]

But \( i_1 = V/Z_1 \), and \( i_2 = V/Z_2 \) so that

\[
\frac{V}{Z_1} + \frac{V}{Z_2} = I \quad \text{or} \quad V = \frac{1}{1/Z_1 + 1/Z_2} I
\]

\[
i_1 = \frac{V}{Z_1} = \frac{1/Z_1}{(1/Z_1 + 1/Z_2)} I = \frac{Y_1 I}{Y_1 + Y_2}
\]

Similarly

\[
i_2 = \frac{Y_2}{Y_1 + Y_2} I.
\]

The current divider may be used to find transfer functions for some simple circuits.

■ Example 7

Find the transfer function

\[ H(s) = \frac{V_o(s)}{I(s)} \]

in the following circuit:

\[ \text{Draw the system as an impedance graph:} \]

\[ \text{\includegraphics{example7}} \]
Let \( Z_1 = 1/sC \), \( Z_2 = R \), and \( Z_3 = sL \). We will use \( V_o(s) = I_2(s)Z_3 \) (at node (b)), and find \( I_2(s) \) from the current division at node (a):

\[
I_2(s) = \frac{1}{Z_2 + Z_3} I(s) = \frac{1}{(1/Z_1)(Z_2 + Z_3) + 1} I(s)
\]

\[
= \frac{1}{Cs(R + Ls) + 1} I(s) = \frac{1}{LCs^2 + RCs + 1} I(s)
\]

\[
V_o(s) = I_2(s)Ls = \frac{Ls}{LCs^2 + RCs + 1} I(s)
\]

or

\[
H(s) = \frac{V_o(s)}{I(s)} = \frac{Ls}{LCs^2 + RCs + 1}
\]